Pricing Asian Options

Lars B. Nielsen

February 2001

Abstract

This Master's Thesis compares selected methods for pricing Asian options. Variance reduction methods in relation to Monte Carlo simulation are also studied. The market model assumed is a basic Black-Scholes model with a lognormal stock price and a deterministic interest rate. Asian options based on an arithmetic average are difficult to price since the distribution of the average is unknown. Consequently, numerical methods and various approximations must be applied.

The pricing methods considered are [Turnbull and Wakeman, 1991], [Levy, 1992], [Milevsky and Posner, 1998], [Vorst, 1992] and last but not least [Curran, 1994]. From these, two new methods are developed and one of them turns out to be surprisingly accurate. Nevertheless, numerical results show that the lower bound suggested by Curran is the most accurate and reliable method for a wide range of parameter values. It is also demonstrated that the price of the geometric Asian option is a very efficient control variate for Monte Carlo simulation of the arithmetic Asian option price.

Keywords: Asian option, average rate option, average price option, exotic option, path-dependent option, arithmetic average, geometric average, pricing bounds, option valuation, Edgeworth series expansion, Wilkinson approximation, conditional expectation, Monte Carlo simulation, control variate, Black-Scholes formula.

Contents

Preface

I have been fascinated by derivatives and in particular options ever since I first came across them. My interest only grew stronger, when I heard the news of the collapse of Barings Bank in 1995. Apparently, an employee in the bank's Singapore division had lost the enormous amount of £850 million by unauthorized trading in the futures and options markets [Leeson, 1996]. In the early 1990s, huge losses were suffered by other derivatives market participants (e.g. Gibson Greetings, Procter and Gamble, Kidder Peabody and Orange County). These losses left the general impression that derivatives were very risky financial products. Though understandable, this is certainly not a fair characterization:

> The stories behind the losses emphasize that derivatives can be used for hedging or speculation; that is, they can be used either to reduce risks or to take risks. The losses were experienced because derivatives were used inappropriately. People who had an implicit or explicit mandate to manage risks prudently decided to take big bets on the future direction of market variables. [Hull, 1997]

Here, at the beginning of a new decade, the interest in derivatives is rapidly increasing and the need for fast and accurate pricing and risk-measuring methods is ubiquitous. I have decided to focus on a small fraction of this ongoing task, namely the pricing of Asian options.

Guide to the chapters

Chapter 1 contains a brief introduction to Asian options followed by the purpose of this thesis. In Chapter 2 I take a look at the Black-Scholes model and present the famous Black-Scholes pricing formula for European call options. The concepts of arbitrage pricing and risk-neutral valuation are also introduced. Chapter 3 gives a more thorough treatment of Asian options and is the foundation of the remaining chapters. Chapter 4 reviews the basics of Monte

Carlo simulation and I study two variance reduction methods, namely the antithetic method and the control variate method. The pricing algorithm in [Turnbull and Wakeman, 1991] is the subject of Chapter 5. The distribution of the arithmetic average is approximated with a lognormal distribution by means of a generalized Edgeworth series expansion. This is a generalization of the method in [Levy, 1992]. In Chapter 6 I study the method suggested by [Milevsky and Posner, 1998], who use the reciprocal gamma distribution to approximate the average. I also combine the methods treated so far to form two new pricing methods. Chapter 7 presents the pricing methods in [Vorst, 1992], which cover both arithmetic and geometric Asian options. The geometric option is used to obtain bounds on the price of the arithmetic option. In Chapter 8 I look at the lower bound developed in [Curran, 1994] and I suggest how to combine this with an upper bound. The main conclusions of the thesis appear from Chapter 9.

How to read this thesis

In order to get the full benefit from the thesis a prior knowledge of mathematics and probability theory is implied. Readers already familiar with the Black-Scholes model and arbitrage pricing can skip Chapter 2 without loss of continuity. From time to time I refer to prices in US dollars but the conclusions apply to any currency. For technical reasons comma has been used as decimal point in the figures, I hope the reader will bear with this small inconsistency. Since it is easier to get a good grasp of multiple graphs when they are in color, all the figures are available at my website

- http://www.lbn.dk/master/
- http://home.imf.au.dk/larsbn/ (Mirror site)

Excel workbooks, Visual Basic code and this thesis in Adobe Portable Document Format (PDF) are also open to the public.

Acknowledgments

I would like to thank my advisor Jørgen Aase Nielsen for helpful suggestions regarding the choice of pricing methods. Allan Mortensen and Mikkel Svenstrup contributed by discussing specific issues with me for which I am grateful. I also owe my thanks to my family and my friends for their encouragement throughout the process. Finally, I am indebted to the woman in my life, Iben Delfs Sørensen, for her love and support.

Chapter 1

Introduction

Asian options are written on the average of the underlying asset over a prespecified period. Consequently, they are also known as average rate or average price options. They were introduced in the oil market in the late 1970s and are now traded in the over-the-counter (OTC) markets all over the world. Asian options are path-dependent due to the averaging and they are often classified as exotic options even though the new generation of exotic options is structurally more complicated.

Compared to standard European options, Asian options have some obvious advantages. First of all, they are often cheaper and better suited for hedging purposes. It is not unusual for exporting companies to receive cashflows in foreign exchange on a regular basis. Buying Asian put options on the exchange rate is a way to ensure that the company realizes at least the average exchange rate over the period in question. Secondly, Asian options reduce the risk of price manipulation near the maturity date, when the underlying is a thinly traded asset or commodity. Asian options are also seen in combination with corporate bonds, which allows the issuing company to share profits, if any, with its bondholders. For this to make sense, the profit of the company should of course depend on the underlying asset of the option.

The downside is that Asian options are hard to price. If the underlying asset is assumed to have a lognormal distribution as in the Black-Scholes model (see Chapter 2), then the arithmetic average does not have a known distribution, since the average is a sum of correlated lognormal random variables. This complicates the quest of a closed-form pricing expression for Asian options similar to the famous Black-Scholes pricing formula for European options.

We shall take a closer look at the Asian option in Chapter 3.

1.1 Purpose of the thesis

The purpose of this thesis is to:

1. Derive and implement selected methods for pricing Asian options.
2. Compare the accuracy of these methods using Monte Carlo simulation as yardstick.

More specifically it is my intention to:

- Study the use of Monte Carlo simulation including variance reduction methods in the context of pricing Asian options.
- Derive the different pricing formulae and all relevant parameters from the Black-Scholes model.
- Implement the methods in Visual Basic for Excel and calculate option prices and bounds for different values of key parameters.
- Compare the results with estimates obtained by means of Monte Carlo simulation to assess the accuracy of the methods.
- Make a comparison of the selected pricing methods with respect to accuracy and computational aspects.

In order to facilitate the implementation I have restrained the task as follows. The starting point is a basic Black-Scholes model with a lognormal stock price, a deterministic interest rate and an equivalent martingale measure. The averaging period is the entire lifetime of the option and the pricing takes place at the initial time. I only price call options since the price of the corresponding put option can be obtained from put-call parity (see e.g. [Bouaziz et al., 1994] or [Vorst, 1992]). Moreover, all options are of European type with a fixed strike and the average is non-weighted, which rule out premature redemption as well as flexible and floating strike Asian options.

I have selected the pricing methods based on their theoretical appeal, expected accuracy and practicability. It was an invariable demand that I was able to do all the programming myself and that the amount of time spent on the programming compared favorably with the benefit. I have excluded finite difference methods among others, because of the time consumption involved, not because of their lack of relevance. Recommendable partial differential equation (PDE) references in this context are [Dewynne and Wilmott, 1995a], [Dewynne and Wilmott, 1995b] and [Rogers and Shi, 1995].

Chapter 2

Preliminaries

Let us start off by taking a look at the famous Black-Scholes model and see how to price a European call option. This approach offers a good deal of intuition, which will come in handy later when we introduce the more complicated Asian option.

2.1 Options

A European option is an asset that gives the holder the right (but not the obligation) to buy or sell an underlying asset (e.g. a stock) at a predetermined date, called the maturity date or the expiration date. The term call option is used to indicate a right to buy, while put option indicates a right to sell. The strike price or the exercise price is the price at which you buy or sell the underlying asset at maturity. Thus, a European option is characterized by its strike price, maturity date, underlying asset and whether it is a call or a put option.

In contrast, American options can be exercised at more than one date, sometimes even continuously throughout the life of the option. As we shall see later, Asian options are options where the exercise price or the stock price at maturity is replaced by an average of the stock prices.

2.2 Black-Scholes model

Let the price of a stock S_t follow the stochastic process

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.1}$$

where μ and σ are constants and W is a standard Brownian motion. Likewise, there is a riskless bond with price process

$$dB_t = rB_t dt \tag{2.2}$$

where r is a constant.

Assumptions

- There are no transaction costs or taxes.
- Short selling in unlimited amounts is allowed.
- All assets are divisible, i.e. you can buy or sell any fraction of the assets.
- The stock pays no dividends.
- There are no arbitrage opportunities.
- Trading takes place continuously.
- The riskless interest rate r is constant and identical for all maturities.

The Black-Scholes model is often called "lognormal", because the stock price process is a Geometric Brownian Motion which implies that S_t has a lognormal distribution (i.e. $\ln S_t$ has a normal distribution). The parameters μ and σ can be interpreted as the "instantaneous" expected rate of return and the "instantaneous" standard deviation of the rate of return. We shall refer to μ as the drift and to σ as the volatility.

2.2.1 Black-Scholes option pricing formula

In their now world famous article [Black and Scholes, 1973], Fischer Black and Myron Scholes gave the following pricing formula for a European call option.

Theorem 1 *The price $C(S_t, t)$ at time t of a European call option on a stock S_t with exercise price K and maturity date T is given by*

$$C(S_t, t) = S_t \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{2.3}$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \tag{2.4}$$

and

$$d_2 = d_1 - \sigma\sqrt{T - t} \tag{2.5}$$

Φ is the standard normal distribution function. σ is the volatility of the stock and r is the riskless interest rate as defined in the Black-Scholes model above.

Proof. See [Black and Scholes, 1973] ■

2.3 Arbitrage pricing

2.3.1 Fundamental PDE

The option pricing formula (2.3) is really the solution to a fundamental partial differential equation (PDE) for pricing derivatives. This PDE is deduced using a technique called arbitrage pricing. The basic idea is to replicate the derivative you want to price by trading continuously in the stock and bond. In the absence of arbitrage, the price of the derivative must be the same as the price of the portfolio of stock and bond.

Theorem 2 *The call option price $C(S_t, t)$ satisfies the PDE*

$$\begin{gathered} -rC(x,t) + C_t(x,t) + rxC_x(x,t) + \frac{1}{2}\sigma^2 x^2 C_{xx}(x,t) = 0, \\ x \in]0, \infty[, \quad t \in [0, T[\end{gathered} \tag{2.6}$$

with boundary condition

$$c(x, T) = (x - K)^+, \quad x \in]0, \infty[\tag{2.7}$$

C_x and C_t are the first order derivatives with respect to the first and second arguments, while C_{xx} is the second order derivative with respect to the first argument.

Proof. See Chapter 6 in [Duffie, 1996]. ■

2.3.2 Risk-neutral valuation

The Black-Scholes formula (2.3) can also be derived using risk-neutral valuation. Even though most investors are risk-averse, we can price derivatives as if all investors were risk-neutral. In their book Financial Calculus [Baxter and Rennie, 1996], Martin Baxter and Andrew Rennie give a brilliant exposition of risk-neutral valuation, including Ito's lemma, equivalent martingale measures, The Martingale representation theorem and Girsanov's theorem. Chapter 3 of the book is a must-read for novices in the area of continuous time finance.

Theorem 3 *The call option price $C(S_t, t)$ is given by*

$$C(S_t, t) = e^{-r(T-t)} E^Q[(S_T - K)^+] \tag{2.8}$$

where Q is the equivalent martingale measure for the discounted stock S_t/B_t and E^Q is expectation under this measure.

Proof. See Chapter 3 in [Baxter and Rennie, 1996]. ■

Chapter 3

Asian options

Asian options are options whose payoff depends on some kind of average of the underlying asset. Stock prices, stock indices, exchange rates, interest rates and commodities are all examples of underlying assets associated with Asian options traded in the over-the-counter market. For purely expositional reasons we assume that the underlying is a dividend-paying stock. The average can be specified in a number of ways and over various time periods, which gives rise to many different options within the Asian class.

3.1 Market model

In the rest of this thesis we shall use the Black-Scholes model of Chapter 2 and the underlying assumptions with a single exception. We assume that the stock pays a continuous dividend rate q, so the market consists of a risky stock S and a riskless bond B, whose prices are given by the stochastic differential equations

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \tag{3.1}$$

$$dB_t = rB_t dt \tag{3.2}$$

r is the riskless interest rate, q is the stock dividend, σ is the volatility of the stock price and W_t is a standard Brownian Motion under an equivalent martingale measure Q. From now on all expectations are with respect to Q. According to [Karatzas and Shreve, 1991] the solutions to (3.1) and (3.2) are

$$S_t = S_0 \exp((r - q - \frac{1}{2}\sigma^2)t + \sigma W_t) \tag{3.3}$$

$$B_t = B_0 \exp(rt) \tag{3.4}$$

From (3.3) we see that the stock price is lognormally distributed.

3.2 Pricing

3.2.1 Averages

As mentioned above, there are many ways to specify the average of the stock prices. Some examples are:
Discrete, arithmetic average

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{3.5}$$

Discrete, geometric average

$$G = (\prod_{i=1}^{n} S_{t_i})^{\frac{1}{n}} \tag{3.6}$$

Continuous, arithmetic average

$$A = \frac{1}{T-t}\int_t^T S_u du \tag{3.7}$$

Continuous, geometric average

$$G = \exp(\frac{1}{T-t}\int_t^T \ln S_u du) \tag{3.8}$$

In the discrete averages there are n prices measured at the time points

$$0 \leq t_1 < t_2 < ... < t_n \leq T \tag{3.9}$$

Usually, these time points are equidistant, e.g. weekly averaging over 1 year. If the average only consists of one price at time T, then we have the European option which is a special case of the Asian option in this respect. The averaging period can be the entire lifetime of the option or some subset of it. In the latter case, the option is called forward-starting.

It is also possible to use a weighted average in which case the Asian option is referred to as flexible. We shall primarily look at the non-weighted, discrete, arithmetic average, but the geometric average plays an important role in some of the pricing methods. Since all traded Asian options are based on discrete averages, the continuous versions are only relevant in theoretical contexts.

3.2.2 Risk-neutral valuation

Only call options are considered, so it is implicit in the rest of this thesis that an option is a call option. The payoff of a fixed strike Asian option with maturity date T and strike price K is the maximum of zero and the difference between the average A and the strike price

$$\max(A - K, 0) \tag{3.10}$$

Using risk-neutral valuation, the arbitrage-free initial price of the option is

$$C = \exp(-rT)E[(A - K)^+] \tag{3.11}$$

where the expectation is with respect to the equivalent martingale measure Q as mentioned above. In contrast, the floating strike Asian option has payoff

$$\max(S_T - A, 0) \tag{3.12}$$

so the initial price is

$$C = \exp(-rT)E[(S_T - A)^+] \tag{3.13}$$

We shall not discuss the floating strike option any further.

The price in (3.11) is not straightforward to calculate, since the distribution of the discrete arithmetic average is unknown. Each of the stock prices has a lognormal distribution, but unfortunately the sum of correlated lognormal random variables is not lognormally distributed. Thus, researchers have developed a number of different numerical methods to obtain the price. These include, but are not limited to, Monte Carlo simulation, approximations of the distribution of the arithmetic average, modifications to the geometric average and conditioning on it, binomial methods, numerical solutions to partial differential equations, Fourier transformation techniques and inversion of non-trivial Laplace transforms. We shall study some of these pricing methods in the following chapters.

Chapter 4

Monte Carlo simulation

Monte Carlo simulation (MCS) is basically a method for evaluating integrals. Often, MCS is neither the best nor the fastest way to solve a given problem, but it is easy to implement and it is applicable to a wide range of problems. In our context, the input for the simulation algorithm is a set of random numbers. The literature on random numbers and random number generators (RNG) is large (see for instance [Glasserman, 1998]), and we shall only note that an ideal RNG produces a sequence of independent, identically distributed (iid.) numbers $u_1, u_2, ...$ In most applications, the uniform distribution on $[0, 1]$ is used.

4.1 Standard Monte Carlo

4.1.1 Linear congruential generator

Since true RNGs are difficult to find, a number of deterministic algorithms have been developed, so-called pseudo-random number generators. We shall use a linear congruential generator of the form

$$x_n = (ax_{n-1} + b) \bmod c, \quad n = 1, 2, ... \tag{4.1}$$

$$u_n = \frac{x_n}{c}, \quad n = 1, 2, ... \tag{4.2}$$

where the seed x_0 is chosen by the user and mod is the modulo function. The u_n's are independent and $U(0, 1)$ distributed. Choosing good values for the parameters a, b and c can have a substantial effect on the outcome of the simulation. [Press et al., 1992] recommends

$$a = 7^5, b = 0, c = 2^{31} - 1 \tag{4.3}$$

c should be large since the algorithm can produce no more than c different numbers. A big advantage of this algorithm is that using the same seed x_0 leads to exactly the same numbers. This makes it useful for comparing results across different methods without storing the numbers.

4.1.2 Inverse distribution function method

We have to draw samples from the (log)normal distribution to simulate the stock prices. Since our random numbers are uniformly distributed we have to somehow transform them into the normal distribution. A very general method for this purpose is the inverse distribution function method. It pairs each of the $U(0,1)$ numbers with the corresponding fractile for the $N(0,1)$ distribution. E.g. 0.5 is transformed into 0 since

$$\Phi^{-1}(0.5) = 0 \tag{4.4}$$

and 0.975 is transformed into 1.96 since

$$\Phi^{-1}(0.975) \approx 1.96 \tag{4.5}$$

Φ^{-1} is the inverse of the standard normal distribution function Φ

$$\Phi^{-1}(x) = \{y \mid \Phi(y) = x\} \tag{4.6}$$

The number $\Phi^{-1}(x)$ is called the x-fractile of the standard normal distribution.

Claim 4 *The numbers transformed by the inverse distribution function method are standard normally distributed.*

Proof. Let $U \sim U(0,1)$. Then $\Phi^{-1}(U) \sim N(0,1)$ since

$$P(\Phi^{-1}(U) \leq x) = P(\Phi(\Phi^{-1}(U)) \leq \Phi(x)) = P(U \leq \Phi(x)) = \Phi(x)$$

■

The function Φ^{-1} can be approximated very closely with a rational function which is convenient for programming. Alternatively, many computer software packages include tools for calculating these values.

4.1.3 Box-Muller method

Box and Muller have proposed a faster method (see [Johnson et al., 1994]) which has been used in our calculations. Let U_1 and U_2 be independent and uniformly distributed over $[0, 1]$. Then

$$X_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2) \tag{4.7}$$
$$X_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2) \tag{4.8}$$

are independent and standard normally distributed.

To sum up, our procedure so far is to generate an iid. sequence of $U(0, 1)$ numbers using the linear congruential generator, and then transform the numbers pairwise to the $N(0, 1)$ distribution using the Box-Muller method.

4.1.4 Sampling

Now, we are ready to sample! We know that the solution to

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \tag{4.9}$$

is

$$S_t = S_0 \exp((r - q - \frac{1}{2}\sigma^2)t + \sigma W_t) \tag{4.10}$$

and that

$$W_t \sim N(0, t) \tag{4.11}$$

so

$$\ln \frac{S_t}{S_0} \sim N((r - q - \frac{1}{2}\sigma^2)t, \sigma^2 t) \tag{4.12}$$

Samples $S_t^{(i)}$ from the distribution of S_t are then obtained by setting

$$S_t^{(i)} = S_0 \exp((r - q - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z^i), \quad i = 1, 2, ..., n \tag{4.13}$$

where $Z^i \sim N(0, 1), \quad i = 1, 2, ..., n, \quad \textit{independent}.$

If we need to sample at different time points, i.e. a path, we can advance the solution as follows

$$S_{t_j}^{(i)} = S_{t_{j-1}}^{(i)} \exp((r - q - \frac{1}{2}\sigma^2)(t_j - t_{j-1}) + \sigma\sqrt{t_j - t_{j-1}}Z^{i,j}),$$
$$j = 1, 2, ..., m, \quad i = 1, 2, ..., n \tag{4.14}$$

where $0 < t_1 < t_2 < ... < t_m = T$. These samples are exact, i.e. there is no approximation error.

4.1.5 Simulation

The strong law of large numbers (see [Hoffmann-Jørgensen, 1994a]) states that

$$\hat{S}_t = \frac{1}{n}\sum_{i=1}^{n} S_t^{(i)} \longrightarrow E[S_t] \quad a.s. \tag{4.15}$$

This is really the heart of Monte Carlo simulation. We approximate the expectation of some random variable by sampling from its distribution and then averaging across replications (samples). $\hat{S}_t$ as defined above is the Monte Carlo estimator of $E[S_t]$. In this simple case, there is no need for MCS since we already know the mean of S_t. We do need MCS in the more general case where we want to evaluate $E[f(S_t)]$ for some function f with the distribution of $f(S_t)$ unknown.

Let $X = (S_{t_1}, S_{t_2}, ..., S_{t_m})$. We would like to evaluate $E[f(X)]$ by Monte Carlo simulation for some continuous function $f : \mathbb{R}^m \to \mathbb{R}$. Let $Y = f(X)$ and let $Y^{(i)}$ be the simulated values of Y obtained by sampling from the distributions of S_{t_j}

$$Y^{(i)} = f(S_{t_1}^{(i)}, S_{t_2}^{(i)}, ..., S_{t_m}^{(i)}), \quad i = 1, 2, ..., n \tag{4.16}$$

The Monte Carlo estimator of $E[Y]$ is

$$\hat{Y} = \frac{1}{n}\sum_{i=1}^{n} Y^{(i)} \tag{4.17}$$

which, by the strong law of large numbers, is converging to the correct mean value

$$\hat{Y} \to E[Y] \quad a.s. \quad as\ n \to \infty \tag{4.18}$$

In other words, the estimator is unbiased. Moreover, the variance of the estimator is

$$Var(\hat{Y}) = Var(\frac{1}{n}\sum_{i=1}^{n} Y^{(i)}) = \frac{1}{n^2}Var(\sum_{i=1}^{n} Y^{(i)}) = \frac{Var(Y)}{n} \tag{4.19}$$

These are two characteristics of a good estimator - convergence towards the correct mean and a variance which decreases as we use more samples.

Now that we have got an estimated price, a natural question would be: "How accurate is it?". We would like to calculate a price interval which contains the true price with an acceptable degree of certainty. The statistical

concepts of significance level and confidence interval can help us here. The significance level α is the borderline probability, which separates events that are significant and non-significant. The implicit assumption is that events with probability less than α do not occur, i.e. they are not significant. In this thesis, we always use $\alpha = 5\%$.
A $(1-\alpha)$ confidence interval is simply an interval which contains the true value behind the estimation with probability $(1-\alpha)$. By the central limit theorem (see [Hoffmann-Jørgensen, 1994a])

$$\frac{\hat{Y} - E[Y]}{\sigma_Y/\sqrt{n}} \to N(0,1) \quad \textit{in distribution} \tag{4.20}$$

where σ_Y is the standard deviation of $Y^{(1)}$, i.e. $\sigma_Y^2 = Var(Y^{(1)}) = Var(Y)$. Since the sample variance s_Y^2 of $Y^{(1)}, Y^{(2)}, ..., Y^{(n)}$

$$s_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y^{(i)} - \hat{Y})^2 \tag{4.21}$$

is converging to σ_Y^2, we can replace σ_Y with s_Y

$$\frac{\hat{Y} - E[Y]}{s_Y/\sqrt{n}} \to N(0,1) \quad \textit{in distribution} \tag{4.22}$$

Proposition 5 *A* $(1-\alpha)$ *confidence interval for* $E[Y]$ *is*

$$[\hat{Y} - z_{1-\alpha/2}\frac{s_Y}{\sqrt{n}},\ \hat{Y} + z_{1-\alpha/2}\frac{s_Y}{\sqrt{n}}] \tag{4.23}$$

where z_α *is the* α*-fractile of the standard normal distribution.*

Proof. From (4.22) we get

$$\begin{aligned}
1-\alpha &= P(z_{\alpha/2} \leq \frac{\hat{Y} - E[Y]}{s_Y/\sqrt{n}} \leq z_{1-\alpha/2}) \\
&= P(z_{\alpha/2}\frac{s_Y}{\sqrt{n}} - \hat{Y} \leq -E[Y] \leq z_{1-\alpha/2}\frac{s_Y}{\sqrt{n}} - \hat{Y}) \\
&= P(\hat{Y} - z_{1-\alpha/2}\frac{s_Y}{\sqrt{n}} \leq E[Y] \leq \hat{Y} - z_{\alpha/2}\frac{s_Y}{\sqrt{n}}) \\
&= P(\hat{Y} - z_{1-\alpha/2}\frac{s_Y}{\sqrt{n}} \leq E[Y] \leq \hat{Y} + z_{1-\alpha/2}\frac{s_Y}{\sqrt{n}})
\end{aligned}$$

The last equation follows from the symmetry of the fractiles $z_{1-\alpha/2} = -z_{\alpha/2}$. We conclude that the true mean value $E[Y]$ is in the above confidence interval with probability $1 - \alpha$. The confidence interval is asymptotically valid as $n \to \infty$, which means that it holds for large values of n. ■

A 95% confidence interval is then

$$[\hat{Y} - 1.96\frac{s_Y}{\sqrt{n}},\ \hat{Y} + 1.96\frac{s_Y}{\sqrt{n}}] \tag{4.24}$$

When reporting estimates, one often includes the sample standard deviation of the estimator $s_Y/\sqrt{n}$. Doubling this amount is a fast way to form the interval (4.24) "mentally", since $2s_Y/\sqrt{n}$ is approximately the halfwidth of the interval.

4.1.6 Method overview

- Generate uniform random numbers.
- Transform the numbers to the standard normal distribution.
- Sample from the distribution of the stock prices.
- Compute discounted Asian option payoffs.
- Obtain an estimate of the option price by averaging across replications.
- Compute the sample standard deviation of the estimator and use it to form a confidence interval.

We would like to shrink the confidence interval as much as possible. This can be done either by reducing the variance or by increasing the number of paths. The latter is very expensive in terms of computer processing time, so we have to look at some variance reduction techniques.

4.2 Variance reduction techniques

4.2.1 Antithetic method

In practice, one often uses 10,000 or 100,000 paths to simulate a price. Depending on the quality of the random numbers, this could lead to results that are not sufficiently precise. Unwanted correlation between the random numbers can render the simulation outcome totally useless. For instance, one can imagine that a set of "bad" parameter values for the linear congruential generator will result in either larger or smaller numbers than expected. We could try to solve this problem by "balancing" each large number with a corresponding small number. This is the intuition behind the antithetic method. Our uniform numbers $U_1, U_2, ...$ can be matched with $1-U_1, 1-U_2, ...$ which are also uniformly distributed over $[0,1]$ and independent. For each path we simulate with $U_1, U_2, ..., U_m$, we generate a second path with $1-U_1, 1-U_2, ..., 1-U_m$.

Formally, let $\theta_1 = g(U_1, U_2, ..., U_m)$ and $\widetilde{\theta}_1 = g(1-U_1, 1-U_2, ..., 1-U_m)$ where g represents the simulation algorithm. We use n pairs of iid. antithetic outcomes $(\theta_i, \widetilde{\theta}_i)\quad i = 1, 2, ..., n$ and a natural unbiased estimator is

$$\widehat{\theta} = \frac{1}{2n}\sum_{i=1}^{n}(\theta_i + \widetilde{\theta}_i) \tag{4.25}$$

Calculating the standard deviation is slightly more difficult than usual since θ_i and $\widetilde{\theta}_i$ are not independent. We define the average of an outcome pair

$$\overline{\theta}_i = \frac{\theta_i + \widetilde{\theta}_i}{2}, \quad i = 1, 2, ..., n \tag{4.26}$$

and we look at $\widehat{\theta}$ as the sample mean of the $\overline{\theta}_i$'s

$$\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\overline{\theta}_i = \frac{1}{n}\sum_{i=1}^{n}\frac{\theta_i + \widetilde{\theta}_i}{2} \tag{4.27}$$

in accordance with (4.25). The sample standard deviation s_θ is now obtained from

$$s_\theta^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\overline{\theta}_i - \widehat{\theta})^2 \tag{4.28}$$

and an asymptotically valid $(1-\alpha)$ confidence interval is

$$[\widehat{\theta} - z_{1-\alpha/2}\frac{s_\theta}{\sqrt{n}},\ \widehat{\theta} + z_{1-\alpha/2}\frac{s_\theta}{\sqrt{n}}] \tag{4.29}$$

An obvious alternative to using antithetic variates is simply to double the number of paths in a standard Monte Carlo simulation. In terms of computational time, these are more or less equivalent since generating random numbers is much faster than sampling (both methods require $2nm$ samples). In order to see which method is better, we compare the variance of the antithetic method

$$\begin{aligned} Var(\overline{\theta}_1) &= Var(\frac{\theta_1 + \widetilde{\theta}_1}{2}) \\ &= \frac{1}{4}Var(\theta_1) + \frac{1}{4}Var(\widetilde{\theta}_1) + \frac{1}{2}Cov(\theta_1, \widetilde{\theta}_1) \\ &= \frac{1}{2}Var(\theta_1) + \frac{1}{2}Cov(\theta_1, \widetilde{\theta}_1) \end{aligned} \tag{4.30}$$

with the variance of a non-antithetic pair

$$\begin{aligned} Var(\frac{\theta_1+\theta_2}{2}) &= \frac{1}{4}Var(\theta_1)+\frac{1}{4}Var(\theta_2)+\frac{1}{2}Cov(\theta_1,\theta_2) \\ &= \frac{1}{2}Var(\theta_1) \end{aligned} \tag{4.31}$$

The conclusion is that if $Cov(\theta_1, \widetilde{\theta}_1) < 0$ then the antithetic method is preferred. [Glasserman, 1998] proves that a sufficient condition for antithetic variables to reduce variance is that the simulation output is monotone in the random numbers. In our case, the option price is increasing in the standard normal random numbers. He also gives a useful tip which applies to sampling from the standard normal distribution. If

$$Z = \Phi^{-1}(U), \quad U \sim U(0,1) \tag{4.32}$$

then $Z \sim N(0,1)$ as already shown and due to symmetry

$$-Z = \Phi^{-1}(1-U) \tag{4.33}$$

This means that a simulation with normal random numbers $Z_1, Z_2, ..., Z_m$ can be expanded with antithetic paths obtained from $-Z_1, -Z_2, ..., -Z_m$. This is applicable even if the Z_i's are not generated from the U_i's by the inverse distribution function method.

4.2.2 Control variates

When making complicated calculations by means of computers, it is always a good idea to perform a calculation manually in a simple setup and compare the result with the output from the programming. Taking this a step further inspires us to use some of the information "hidden" in our calculations so far to improve our estimates. We could, for instance, price a standard European call option along with the Asian option and compare the outcome with the theoretical price obtained from the Black-Scholes formula. If we observed significant errors, we could correct the estimates with this new information. This is basically the control variates approach to reducing variance and thereby improving the simulation results.

More formally, the control variate is a random variable which depends on the simulation input and has a high degree of correlation with the variable we want to estimate. Moreover, we must be able to calculate the expectation of the control variate.

As before, let $X = (S_{t_1}, S_{t_2}, ..., S_{t_m})$ and $Y = f(X)$. We use the function h to represent the calculation of the control variate W, so

$$W = h(X) \tag{4.34}$$

The simulated values of W are

$$W^{(i)} = h(S_{t_1}^{(i)}, S_{t_2}^{(i)}, ..., S_{t_m}^{(i)}), \quad i = 1, 2, ..., n \tag{4.35}$$

The new estimator is

$$\begin{aligned} \hat{Y}_{cv} &= \frac{1}{n}\sum_{i=1}^{n} Y^{(i)} - (W^{(i)} - E[W]) \\ &= \frac{1}{n}\sum_{i=1}^{n} Y^{(i)} - W^{(i)} + E[W] \end{aligned} \tag{4.36}$$

The strong law of large numbers tells us that the estimator is unbiased, and the central limit theorem gives us a $(1-\alpha)$ confidence interval

$$[\hat{Y}_{cv} - z_{1-\alpha/2}\frac{s_{cv}}{\sqrt{n}},\ \hat{Y}_{cv} + z_{1-\alpha/2}\frac{s_{cv}}{\sqrt{n}}] \tag{4.37}$$

where s_{cv} is the sample standard deviation obtained from

$$s_{cv}^2 = \frac{1}{n-1}\sum_{i=1}^{n}((Y^{(i)} - W^{(i)} + E[W]) - \hat{Y}_{cv})^2 \tag{4.38}$$

The method can be further improved by using the estimator

$$\hat{Y}_{cvi} = \frac{1}{n}\sum_{i=1}^{n} Y^{(i)} - \beta(W^{(i)} - E[W]) \tag{4.39}$$

for some β. The details can be found in [Glasserman, 1998]. The downside is that more paths are needed to approximate the optional β.

Geometric option

We mentioned earlier that the control variable must have a high degree of correlation with the variable we want to estimate. This is indeed the case when pricing Asian arithmetic options with geometric options as control variates. Furthermore, the geometric average is lognormally distributed, which makes it possible to give a closed form expression for the geometric option price. The discrete geometric average G is

$$G = (\prod_{j=1}^{m} S_{t_j})^{1/m} \tag{4.40}$$

so the price of the geometric option is according to Chapter 7

$$C_G = \exp(-rT)\{\exp(\mu_G + \frac{1}{2}\sigma_G^2)\Phi(d_1) - K\Phi(d_2)\} \tag{4.41}$$

with

$$\mu_G = \ln S_0 + (r - q - \frac{1}{2}\sigma^2)\frac{T+h}{2} \tag{4.42}$$

$$\sigma_G^2 = \sigma^2 h \frac{(2n+1)(n+1)}{6n} \tag{4.43}$$

and

$$d_1 = \frac{\mu_G - \ln K + \sigma_G^2}{\sigma_G} \tag{4.44}$$

$$d_2 = d_1 - \sigma_G \tag{4.45}$$

4.3 Numerical results

Here, we present the results of using Monte Carlo simulation to price Asian options. We also explain in detail how the different averages, prices etc. have been calculated, and the Visual Basic code behind it all appears from Appendix A. In addition to standard Monte Carlo simulation, we have applied the antithetic method as well as control variates to see which method performs best.

The price of the Asian option is

$$C = \exp(-rT)E[\max(A - K, 0)] \tag{4.46}$$

where A is the discrete arithmetic average

$$A = \frac{1}{m}\sum_{j=1}^{m} S_{t_j} \tag{4.47}$$

We use equidistant time points, so

$$t_j = t_{j-1} + h, \quad h = \frac{T}{m}, \quad t_0 = 0, \quad j = 1, 2, ..., m \tag{4.48}$$

The sample averages are

$$A^{(i)} = \frac{1}{m}\sum_{j=1}^{m} S_{t_j}^{(i)}, \quad i = 1, 2, ..., n \tag{4.49}$$

with $S_{t_j}^{(i)}$ given by (4.14).

4.3.1 Standard Monte Carlo

Sample option prices are calculated as

$$C^{(i)} = \exp(-rT)\max(A^{(i)} - K, 0), \quad i = 1, 2, ..., n \tag{4.50}$$

which leads us to the Monte Carlo estimator

$$\hat{C} = \frac{1}{n}\sum_{i=1}^{n} C^{(i)} \tag{4.51}$$

The sample variance s_C^2 is

$$s_C^2 = \frac{1}{n-1}\sum_{i=1}^{n}(C^{(i)} - \hat{C})^2 \tag{4.52}$$

and a 95% confidence interval is

$$[\hat{C} - 1.96\frac{s_C}{\sqrt{n}},\ \hat{C} + 1.96\frac{s_C}{\sqrt{n}}] \tag{4.53}$$

Table 4.1 and Table 4.2 contain option prices and standard deviations for different values of the stock price volatility σ and the maturity T of the option. The other parameters of interest have been fixed at

$$K = 100, \quad m = 12, \quad n = 50000, \quad q = 0, \quad r = 0.05, \quad S_0 = 100 \tag{4.54}$$

which means that both the initial stock price and the strike price are 100, the number of prices in the average is 12 (corresponding to monthly averaging when $T = 1$), the stock pays no dividends, the interest rate is 5% and we have simulated with 50,000 paths. The last three columns in the tables represent the simulated price of a European option, the standard deviation of the estimator and the theoretical Black-Scholes value. The European option has been included to give us an idea of the accuracy of the Monte Carlo estimates. If the simulated European prices are significantly different from the theoretical prices, we should expect that the Asian prices are also imprecise.

The Monte Carlo estimator of the European option price is

$$\hat{C}_E = \frac{1}{n}\sum_{i=1}^{n} C_E^{(i)} \tag{4.55}$$

where the samples are

$$C_E^{(i)} = \exp(-rT)\max(S_{t_m}^{(i)} - K, 0), \quad i = 1, 2, ..., n \tag{4.56}$$

The sample variance s_E^2 is

$$s_E^2 = \frac{1}{n-1}\sum_{i=1}^{n}(C_E^{(i)} - \hat{C}_E)^2 \tag{4.57}$$

and a 95% confidence interval is

$$[\hat{C}_E - 1.96\frac{s_E}{\sqrt{n}},\ \hat{C}_E + 1.96\frac{s_E}{\sqrt{n}}] \tag{4.58}$$

The theoretical price C_E is obtained from (2.3).

It is obvious from Table 4.1 that the standard deviations are too large to be of any good. As we would expect, both the option price and the standard deviation are increasing in the stock price volatility. This can be seen in Figure 4.1 where we also see that the spread between the European and Asian prices increase as volatility increases. The Asian price is lower than the European price mainly because the averaging tends to lower the variance of the terminal payoff, and hence lower the option price. Figure 4.1 also shows the endpoints of the 95% confidence interval for $\hat{C}_E$ as well as the Black-Scholes price. $\hat{C}_E$ has been subtracted from all values to make the graphs distinguishable from each other. The true price is inside the confidence interval, but the interval is so wide that we really can't use our estimates to anything. As can be seen, the Monte Carlo simulation consistently underprices the European option, which could indicate that our random numbers are not as uncorrelated as we would wish.

Table 4.2 is similar to Table 4.1 but this time we have priced options with maturities from half a year up to five years. It is evident that both prices and standard deviations increase as maturity increases. This is also illustrated in Figure 4.2. The confidence intervals are too wide for our purpose, so we could consider increasing the number of paths. From (4.58) we see that doubling the precision requires four times as many paths. If we demand that the confidence interval is no wider than 0.01, i.e. 1 cent, then we need more than 250 million paths to price the option with σ=0.50 in Table 4.1. This would probably take forever on a standard PC, so variance reduction methods are indeed necessary.

σ	T	$\hat{C}$	$s_C/\sqrt{n}$	$\hat{C}_E$	$s_E/\sqrt{n}$	C_E
0.05	1	2.92	0.0114	5.28	0.0196	5.28
0.10	1	3.89	0.0200	6.80	0.0344	6.80
0.15	1	4.99	0.0287	8.59	0.0495	8.59
0.20	1	6.13	0.0377	10.44	0.0654	10.45
0.25	1	7.28	0.0471	12.32	0.0821	12.34
0.30	1	8.43	0.0568	14.20	0.0998	14.23
0.35	1	9.58	0.0670	16.09	0.1185	16.13
0.40	1	10.73	0.0776	17.97	0.1383	18.02
0.45	1	11.88	0.0886	19.84	0.1593	19.91
0.50	1	13.03	0.1001	21.70	0.1815	21.79

Table 4.1: Monte Carlo option prices for different volatilities. K=100, m=12, n=50000, q=0, r=0.05, S_0=100.

σ	T	$\hat{C}$	$s_C/\sqrt{n}$	$\hat{C}_E$	$s_E/\sqrt{n}$	C_E
0.20	**0.5**	4.09	0.0256	6.88	0.0435	6.89
0.20	**1.0**	6.13	0.0377	10.44	0.0654	10.45
0.20	**1.5**	7.80	0.0476	13.43	0.0837	13.44
0.20	**2.0**	9.28	0.0561	16.10	0.1000	16.13
0.20	**2.5**	10.62	0.0638	18.57	0.1151	18.60
0.20	**3.0**	11.85	0.0708	20.89	0.1293	20.92
0.20	**3.5**	13.00	0.0773	23.08	0.1428	23.12
0.20	**4.0**	14.08	0.0835	25.17	0.1558	25.21
0.20	**4.5**	15.09	0.0892	27.16	0.1683	27.22
0.20	**5.0**	16.05	0.0947	29.08	0.1804	29.14

Table 4.2: Monte Carlo option prices for different maturities. K=100, m=12, n=50000, q=0, r=0.05, S_0=100.

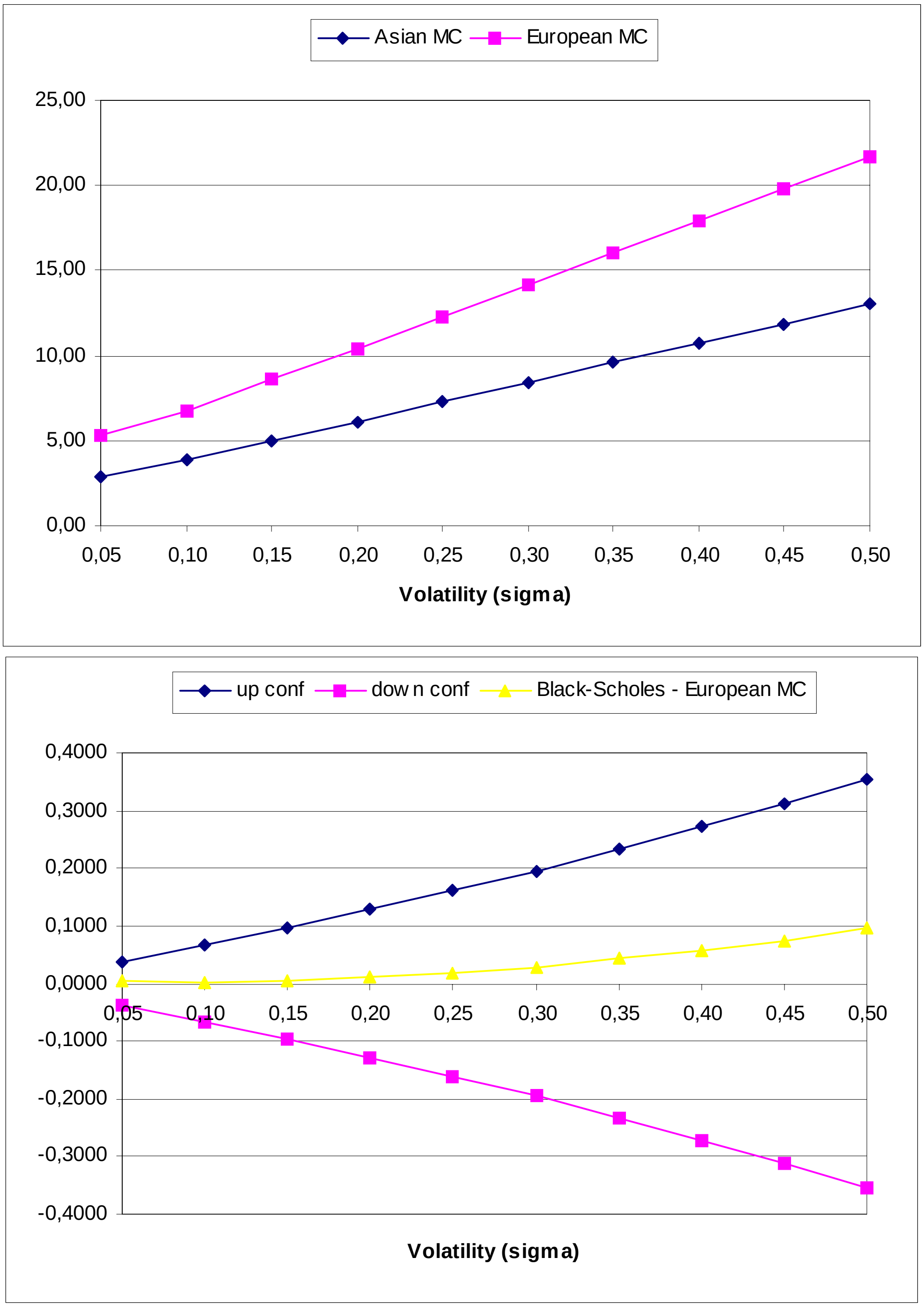


Figure 4.1: Top: Monte Carlo estimates for Asian and European option prices. Bottom: 95% confidence intervals and theoretical European option prices. All values subtracted by Monte Carlo estimate.

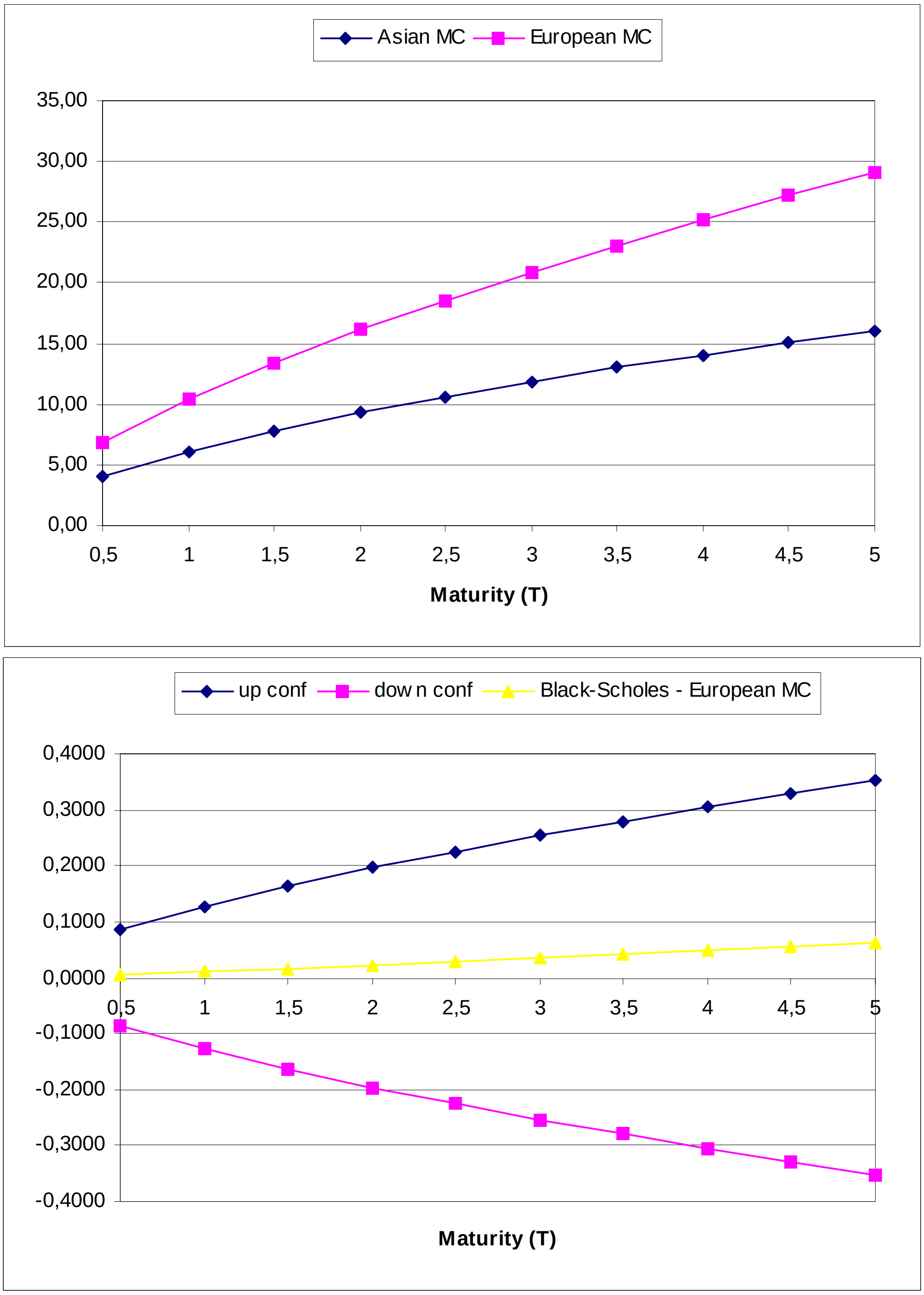


Figure 4.2: Top: Monte Carlo estimates for Asian and European option prices. Bottom: 95% confidence intervals and theoretical European option prices. All values subtracted by Monte Carlo estimate.

4.3.2 Antithetic method

For each $C^{(i)}$ in (4.50) we generate a $\widetilde{C}^{(i)}$ by using $\widetilde{Z}^{i,j} = -Z^{i,j}, \quad j = 1, 2, ..., m$ to simulate the stock prices. The Monte Carlo estimator is

$$\hat{C} = \frac{1}{n} \sum_{i=1}^{n} \frac{C^{(i)} + \widetilde{C}^{(i)}}{2} \tag{4.59}$$

and a 95% confidence interval is

$$[\hat{C} - 1.96\frac{s_C}{\sqrt{n}}, \ \hat{C} + 1.96\frac{s_C}{\sqrt{n}}] \tag{4.60}$$

with

$$s_C^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\frac{C^{(i)} + \widetilde{C}^{(i)}}{2} - \hat{C})^2 \tag{4.61}$$

The European option prices are simulated in a similar fashion.

Table 4.3 and Table 4.4 contain simulated option prices for different volatilities and maturities. The same remarks as for the standard method above apply. It is worth noting that the Asian prices have increased by up to 7 cents (see also Figure 4.3) whereas the European prices only have increased by up to 2 cents. Since the standard method underpriced the European options, it might have underpriced the Asian options as well, so we expect that the higher antithetic estimates of the Asian prices are better. Even though the antithetic method has reduced the standard deviations by a factor two, we are still not satisfied with the accuracy. We postpone our final conclusion regarding the antithetic method until we have seen how the other variance reduction method performs.

4.3.3 Control variates

In addition to (4.49), the sample geometric averages are

$$G^{(i)} = (\prod_{j=1}^{m} S_{t_j}^{(i)})^{1/m}, \quad i = 1, 2, ..., n \tag{4.62}$$

and sample geometric option prices are calculated as

$$C_G^{(i)} = \exp(-rT) \max(G^{(i)} - K, 0), \quad i = 1, 2, ..., n \tag{4.63}$$

σ	**T**	$\hat{\mathbf{C}}$	$\mathbf{s}_C/\sqrt{n}$	$\hat{\mathbf{C}}_E$	$\mathbf{s}_E/\sqrt{n}$	$\mathbf{C}_E$
0.05	1	2.93	0.0028	5.28	0.0044	5.28
0.10	1	3.90	0.0079	6.80	0.0133	6.80
0.15	1	5.01	0.0131	8.59	0.0226	8.59
0.20	1	6.15	0.0186	10.44	0.0325	10.45
0.25	1	7.31	0.0243	12.32	0.0431	12.34
0.30	1	8.47	0.0303	14.21	0.0545	14.23
0.35	1	9.63	0.0367	16.10	0.0666	16.13
0.40	1	10.79	0.0434	17.98	0.0797	18.02
0.45	1	11.95	0.0506	19.85	0.0936	19.91
0.50	1	13.10	0.0581	21.72	0.1086	21.79

Table 4.3: Monte Carlo option prices for different volatilities. Antithetic method. K=100, m=12, n=50000, q=0, r=0.05, S_0=100.

σ	**T**	$\hat{\mathbf{C}}$	$\mathbf{s}_C/\sqrt{n}$	$\hat{\mathbf{C}}_E$	$\mathbf{s}_E/\sqrt{n}$	$\mathbf{C}_E$
0.20	**0.5**	4.11	0.0127	6.88	0.0218	6.89
0.20	**1.0**	6.15	0.0186	10.44	0.0325	10.45
0.20	**1.5**	7.83	0.0232	13.43	0.0414	13.44
0.20	**2.0**	9.31	0.0272	16.11	0.0495	16.13
0.20	**2.5**	10.66	0.0307	18.58	0.0569	18.60
0.20	**3.0**	11.89	0.0340	20.89	0.0640	20.92
0.20	**3.5**	13.05	0.0370	23.08	0.0708	23.12
0.20	**4.0**	14.13	0.0399	25.17	0.0774	25.21
0.20	**4.5**	15.15	0.0426	27.16	0.0838	27.22
0.20	**5.0**	16.12	0.0452	29.08	0.0901	29.14

Table 4.4: Monte Carlo option prices for different maturities. Antithetic method. K=100, m=12, n=50000, q=0, r=0.05, S_0=100.

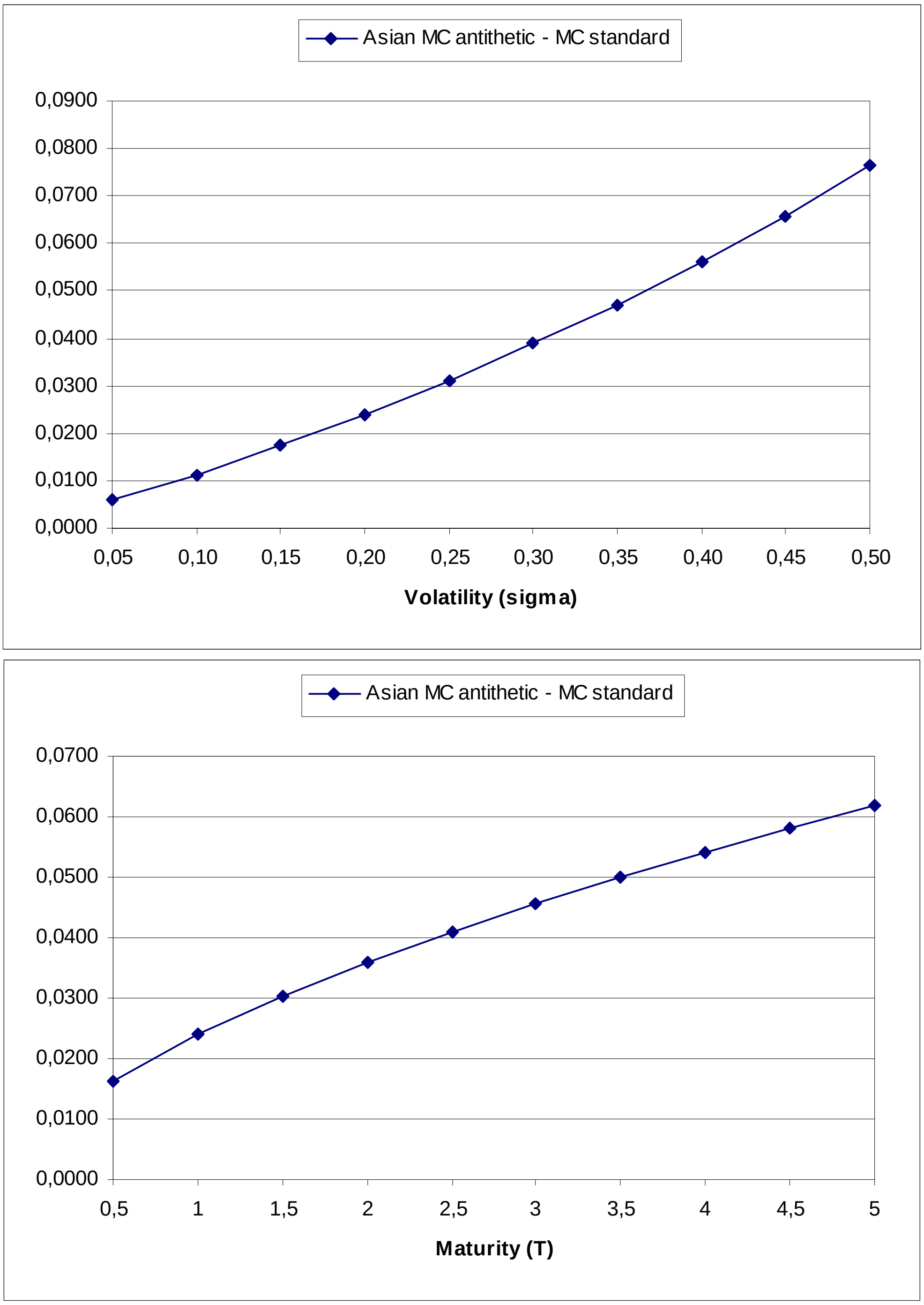


Figure 4.3: Difference between Monte Carlo estimates with and without variance reduction.

whereas the true geometric option price C_G is given by (4.41).
The Monte Carlo estimator of the Asian option price is then

$$\hat{C} = \frac{1}{n}\sum_{i=1}^{n} C^{(i)} - C_G^{(i)} + C_G \tag{4.64}$$

and a 95% confidence interval is

$$[\hat{C} - 1.96\frac{s_C}{\sqrt{n}},\ \hat{C} + 1.96\frac{s_C}{\sqrt{n}}] \tag{4.65}$$

with

$$s_C^2 = \frac{1}{n-1}\sum_{i=1}^{n}(C^{(i)} - C_G^{(i)} + C_G - \hat{C})^2 \tag{4.66}$$

Likewise, the Monte Carlo estimator of the geometric option price is

$$\hat{C}_G = \frac{1}{n}\sum_{i=1}^{n} C_G^{(i)} \tag{4.67}$$

The sample variance is

$$s_G^2 = \frac{1}{n-1}\sum_{i=1}^{n}(C_G^{(i)} - \hat{C}_G)^2 \tag{4.68}$$

so a 95% confidence interval is

$$[\hat{C}_G - 1.96\frac{s_G}{\sqrt{n}},\ \hat{C}_G + 1.96\frac{s_G}{\sqrt{n}}] \tag{4.69}$$

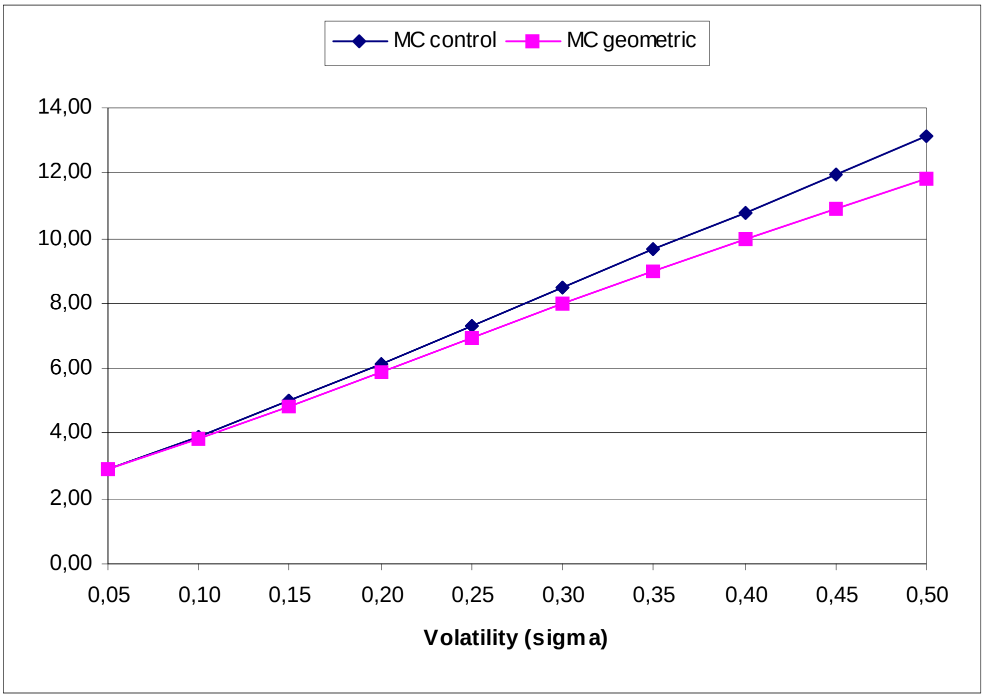
Table 4.5 and Table 4.6 show simulated option prices for the Asian option and the geometric option as well as the theoretical geometric price. The prices are close for small volatilities and small maturities (see Figure 4.4 and Figure 4.5). The geometric options are cheaper than the Asian options because the geometric average is always lower than the arithmetic average. The simulation has consistently underpriced the geometric option, which again could indicate correlation between the random numbers. The standard deviations of $\hat{C}$ have been reduced by a factor 9 or more which has dramatically narrowed the confidence intervals as seen in Figure 4.4 and Figure 4.5. From these it is also evident that the estimates from the antithetic method and the control variates method are almost the same.

$\boldsymbol{\sigma}$	T	$\hat{\mathbf{C}}$	$s_C/\sqrt{n}$	$\hat{\mathbf{C}}_G$	$s_G/\sqrt{n}$	$\mathbf{C}_G$
0.05	1	2.93	0.0001	2.89	0.0113	2.90
0.10	1	3.90	0.0004	3.82	0.0197	3.84
0.15	1	5.01	0.0009	4.86	0.0281	4.88
0.20	1	6.16	0.0016	5.91	0.0366	5.94
0.25	1	7.31	0.0024	6.96	0.0453	6.99
0.30	1	8.47	0.0035	7.98	0.0542	8.02
0.35	1	9.64	0.0049	8.99	0.0632	9.04
0.40	1	10.80	0.0065	9.97	0.0725	10.03
0.45	1	11.96	0.0085	10.92	0.0820	11.00
0.50	1	13.11	0.0107	11.85	0.0917	11.94

Table 4.5: Monte Carlo option prices for different volatilities. Control variates. K=100, m=12, n=50000, q=0, r=0.05, S_0=100.

$\boldsymbol{\sigma}$	T	$\hat{\mathbf{C}}$	$s_C/\sqrt{n}$	$\hat{\mathbf{C}}_G$	$s_G/\sqrt{n}$	$\mathbf{C}_G$
0.20	**0.5**	4.11	0.0007	3.99	0.0250	4.01
0.20	**1.0**	6.16	0.0016	5.91	0.0366	5.94
0.20	**1.5**	7.84	0.0025	7.47	0.0457	7.50
0.20	**2.0**	9.32	0.0034	8.81	0.0535	8.85
0.20	**2.5**	10.66	0.0045	10.01	0.0604	10.05
0.20	**3.0**	11.90	0.0055	11.10	0.0666	11.15
0.20	**3.5**	13.05	0.0066	12.10	0.0722	12.16
0.20	**4.0**	14.14	0.0078	13.03	0.0774	13.09
0.20	**4.5**	15.16	0.0090	13.90	0.0822	13.96
0.20	**5.0**	16.12	0.0102	14.71	0.0867	14.77

Table 4.6: Monte Carlo option prices for different maturities. Control variates. K=100, m=12, n=50000, q=0, r=0.05, S_0=100.



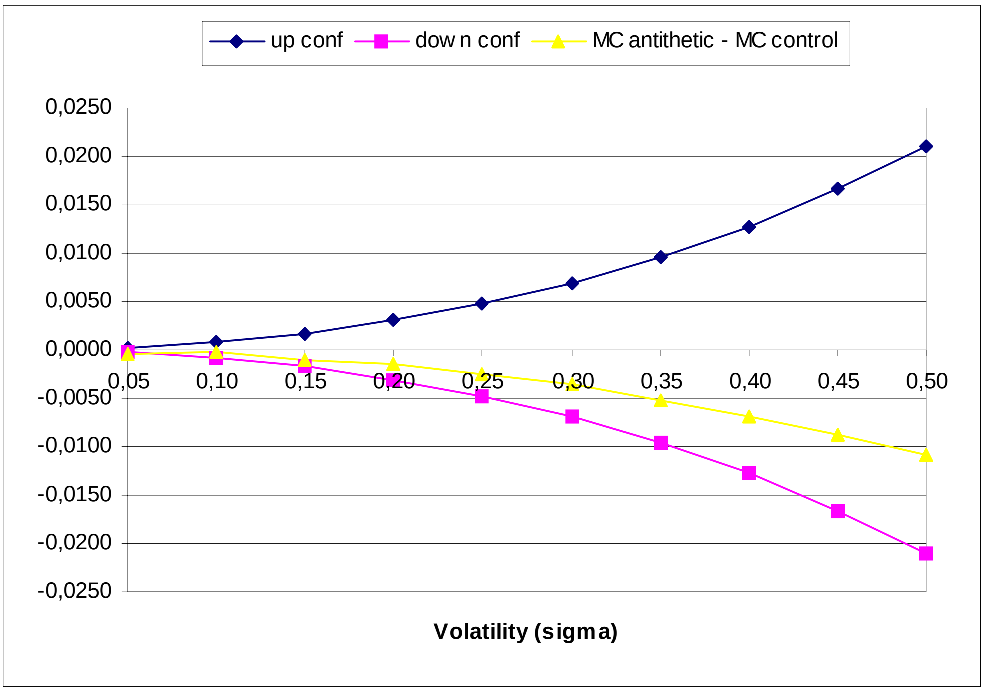


Figure 4.4: Top: Monte Carlo option prices of Asian options with arithmetic and geometric averages. Control variates. Bottom: 95% confidence intervals (estimate subtracted) and difference between antithetic method and control variates.

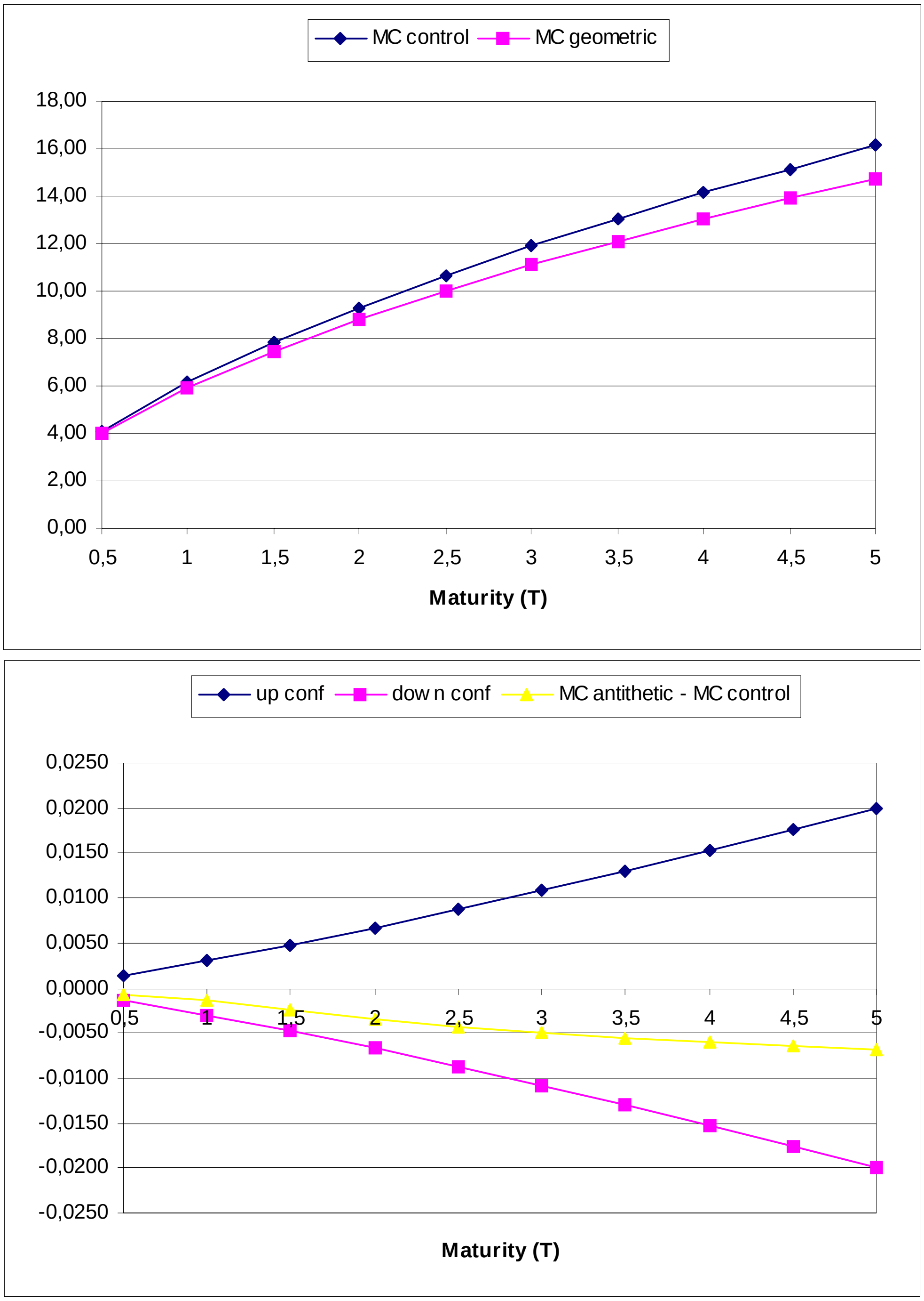


Figure 4.5: Top: Monte Carlo option prices of Asian options with arithmetic and geometric averages. Control variates. Bottom: 95% confidence intervals (estimate subtracted) and difference between antithetic method and control variates.

4.3.4 Conclusion

We have used Monte Carlo simulation to price various Asian and European options. From the simulations we conclude that the standard method is not accurate enough and that increasing the number of paths is not computationally feasible. This is in accordance with the results in [Kemna and Vorst, 1990]. The random numbers generated from the linear congruential generator and transformed by the Box-Muller method showed signs of correlation, so using more sophisticated methods to generate random numbers could prove advantageous. We applied two different variance reduction methods which both gave very good estimates, but the antithetic method did not reduce the standard deviations to an acceptable level. In contrast, using the Asian option based on a geometric average as a control variate for the arithmetic Asian option turned out extremely well due to the strong correlation between the two kinds of averages. We obtained 95% confidence intervals only a few cents wide even for large volatilities and large maturities.

Chapter 5

Turnbull and Wakeman algorithm

[Turnbull and Wakeman, 1991] price Asian options by approximating the distribution of the arithmetic average with a lognormal distribution. Similar methods have been used by [Levy, 1992] and others, but Turnbull and Wakeman use a generalized Edgeworth series expansion which also takes differences in skewness and kurtosis into account. This approach relies on the fact that all the moments of the arithmetic average can easily be calculated even though the distribution of the average is unknown. The algorithm is very fast and, as we shall se, it is relatively precise.

5.1 Generalized Edgeworth series expansion

The generalized Edgeworth series expansion is a technique for approximating a distribution function F (the true distribution) with an alternative distribution function G (the approximating distribution). It has similarities to the well-known Taylor series expansion of an analytical function. We assume that F and G have continuous density functions f and g and that the random variable X has distribution F.

Define the m'th moment

$$\begin{aligned}\alpha_m(F) &= \int_{-\infty}^{\infty} x^m f(x)dx \\ &= E[X^m], \qquad m = 1, 2, ... \qquad (5.1)\end{aligned}$$

and the m'th central moment

$$\begin{aligned} \mu_m(F) &= \int_{-\infty}^{\infty} (x - \alpha_1(F))^m f(x) dx \\ &= E[(X - E[X])^m], \qquad m = 1, 2, ... \end{aligned} \tag{5.2}$$

and the first four cumulants

$$\kappa_1(F) = \alpha_1(F) = E[X] \tag{5.3}$$
$$\kappa_2(F) = \mu_2(F) = E[(X - E[X])^2] \tag{5.4}$$
$$\kappa_3(F) = \mu_3(F) = E[(X - E[X])^3] \tag{5.5}$$
$$\begin{aligned} \kappa_4(F) &= \mu_4(F) - 3(\mu_2(F))^2 \\ &= E[(X - E[X])^4] - 3(E[(X - E[X])^2])^2 \end{aligned} \tag{5.6}$$

κ_1 is the mean, κ_2 is the variance, κ_3 is a measure of skewness and κ_4 is a measure of kurtosis. We define $\alpha_m(G)$, $\mu_m(G)$ and $\kappa_m(G)$ likewise as above.

Theorem 6 *Assume that the first five moments of both distributions exist and that G is five times differentiable. We can then expand $f(x)$ in terms of $g(x)$ as follows*

$$\begin{aligned} f(x) &= g(x) + \frac{\kappa_2(F) - \kappa_2(G)}{2!} \frac{d^2 g}{dx^2}(x) - \frac{\kappa_3(F) - \kappa_3(G)}{3!} \frac{d^3 g}{dx^3}(x) \\ &\quad + \frac{\kappa_4(F) - \kappa_4(G) + 3(\kappa_2(F) - \kappa_2(G))^2}{4!} \frac{d^4 g}{dx^4}(x) + \varepsilon(x) \end{aligned} \tag{5.7}$$

where $\varepsilon(x)$ is a residual error term and $\kappa_1(G)$ has been set equal to $\kappa_1(F)$.

Proof. See the Appendix in [Jarrow and Rudd, 1982]. ■

Hence it appears that the difference between $f(x)$ and $g(x)$ is expressed as three terms plus an error term. The first term adjusts for difference in variance between the two distributions, while the second and third terms adjust for difference in skewness and kurtosis. The weighting factors are respectively the second, third and fourth derivatives of g with respect to x.

The magnitude of the error term is of course very interesting but unfortunately nothing can be said about this in general. It seems plausible though that the more cumulants we include, the better the approximation is. We shall take a closer look at this problem in Section 5.3.

5.2 Algorithm

5.2.1 Model

As before, the stock price is given by

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t \tag{5.8}$$

and the arithmetic average is

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{5.9}$$

where the averaging dates are

$$t_i = ih, \quad h = \frac{T}{n}, \quad i = 1, 2, ..., n \tag{5.10}$$

The option payoff at maturity T is $\max\{A-K, 0\}$, so the option price at time 0 is

$$\begin{aligned} C &= \exp(-rT)E[(A-K)^+] \\ &= \exp(-rT)\int_{-\infty}^{\infty}(x-K)^+ f_A(x)dx \\ &= \exp(-rT)\int_{K}^{\infty}(x-K) f_A(x)dx \end{aligned} \tag{5.11}$$

where f_A is the true, but unknown, density function of A. We could now substitute $f(x)$ from (5.7) for $f_A(x)$ in (5.11), but for technical reasons we shall first rewrite the average.

Define the relative stock price from time t_{i-1} to t_i

$$R_i = \frac{S_{t_i}}{S_{t_{i-1}}}, \quad i = 1, 2, ..., n, \quad t_0 = 0 \tag{5.12}$$

and define

$$L_{n+1} = 1 \tag{5.13}$$

$$L_i = 1 + R_i L_{i+1}, \quad i = 2, 3, ..., n \tag{5.14}$$

Now, we can write

$$S_{t_i} = S_{t_{i-1}} R_i = S_{t_1} R_2 ... R_i, \quad i = 2, 3, ..., n \tag{5.15}$$

and the average can be written as

$$\begin{aligned} A &= \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \\ &= \frac{1}{n}(S_{t_1} + S_{t_1}R_2 + S_{t_1}R_2R_3 + ... + S_{t_1}R_2...R_n) \\ &= \frac{S_{t_1}}{n}(1 + R_2 + R_2R_3 + ... + R_2...R_n) \\ &= \frac{S_{t_1}}{n}(1 + R_2(1 + R_3(1 + ...(1 + R_n)))) \\ &= \frac{S_{t_1}}{n}L_2 \\ &= \frac{S_0}{n}R_1L_2 \end{aligned} \tag{5.16}$$

We want to get rid of the deterministic constant, so we define the scaled average Y

$$Y = \frac{n}{S_0}A = R_1L_2 \tag{5.17}$$

and the scaled strike k

$$k = \frac{n}{S_0}K \tag{5.18}$$

5.2.2 Moments and cumulants of the true distribution

In order to apply the Edgeworth series expansion (5.7) to the distribution of Y, we have to calculate the cumulants of Y which, in turn, requires us to calculate the moments of Y. From (5.8), we know that

$$S_{t_i} = S_{t_{i-1}} \exp((r - q - \frac{1}{2}\sigma^2)h + \sigma(W_{t_i} - W_{t_{i-1}})), \quad i = 1, 2, ..., n \tag{5.19}$$

so R_i is lognormally distributed

$$\log R_i \sim N((r - q - \frac{1}{2}\sigma^2)h, \sigma^2 h) \tag{5.20}$$

and its moments are

$$E[R_i^m] = \exp((r - q - \frac{1}{2}\sigma^2)hm + \frac{1}{2}\sigma^2 hm^2), \quad m = 1, 2, ... \tag{5.21}$$

As the Brownian Motion has independent increments, R_1 and L_2 are independent, and

$$E[Y^m] = E[(R_1 L_2)^m] = E[R_1^m]E[L_2^m], \quad m = 1, 2, ... \tag{5.22}$$

From (5.14) we get

$$E[L_i^m] = E[(1 + R_i L_{i+1})^m], \quad i = 2, 3, ..., n, \quad m = 1, 2, ...$$

As shown in Appendix B, the first four moments of L_i are

$$\begin{aligned}
E[L_i] &= E[1 + R_i L_{i+1}] && (5.23)\\
E[L_i^2] &= 1 + E[R_i^2]E[L_{i+1}^2] + 2E[R_i]E[L_{i+1}] && (5.24)\\
E[L_i^3] &= 1 + E[R_i^3]E[L_{i+1}^3] + 3E[R_i^2]E[L_{i+1}^2] + 3E[R_i]E[L_{i+1}] && (5.25)\\
E[L_i^4] &= 1 + E[R_i^4]E[L_{i+1}^4] + 4E[R_i^3]E[L_{i+1}^3] + 6E[R_i^2]E[L_{i+1}^2]\\
&\quad +4E[R_i]E[L_{i+1}] && (5.26)
\end{aligned}$$

By definition of L_i, it is necessary to calculate these moments recursively starting with $i = n$ and ending with $i = 2$.

The cumulants can be calculated from the moments as follows

$$\begin{aligned}
\kappa_1(F) &= E[Y] && (5.27)\\
\kappa_2(F) &= E[Y^2] - E[Y]^2 && (5.28)\\
\kappa_3(F) &= E[Y^3] + 2E[Y]^3 - 3E[Y^2]E[Y] && (5.29)\\
\kappa_4(F) &= E[Y^4] - 6E[Y]^4 - 4E[Y^3]E[Y]\\
&\quad +12E[Y^2]E[Y]^2 - 3E[Y^2]^2 && (5.30)
\end{aligned}$$

The details can be found in Appendix B.

5.2.3 Approximating distribution

[Turnbull and Wakeman, 1991] have chosen the lognormal distribution as approximating distribution due to its widespread use in modern finance. By construction, the first moment of the approximating distribution G is set equal to the first moment of the true distribution F, cf. Theorem 6. The lognormal distribution is characterized by two parameters, so we need another equation to "calibrate" the distribution. There are several ways to do this, one of which is to set the second moments of the two distributions equal, i.e.

$$\alpha_2(G) = \alpha_2(F) \tag{5.31}$$

Since $\alpha_1(G) = \alpha_1(F)$, we can also express the above in terms of the cumulants

$$\kappa_2(G) = \kappa_2(F) \tag{5.32}$$

Intuitively, this should result in a good fit to the true distribution. We expect that matching the first two moments and adjusting for some of the higher moments will explain a substantial part of the influence which the distribution of the average has on the option price.

We have already seen how to calculate the moments of the true distribution, and the moments of the lognormal distribution with parameters ν and λ^2 are

$$\alpha_m(G) = \int_{-\infty}^{\infty} x^m g(x)dx = \exp(\nu m + \frac{1}{2}\lambda^2 m^2), \quad m = 1, 2, ... \tag{5.33}$$

To find the parameters of the approximating distribution, we match the first two moments and solve the following system of equations

$$\alpha_1(F) = \exp(\nu + \frac{1}{2}\lambda^2) \tag{5.34}$$

$$\alpha_2(F) = \exp(2\nu + 2\lambda^2) \tag{5.35}$$

As shown in Appendix B, the solution is

$$\nu = 2\ln\alpha_1(F) - \frac{1}{2}\ln\alpha_2(F) \tag{5.36}$$

$$\lambda^2 = \ln\alpha_2(F) - 2\ln\alpha_1(F) \tag{5.37}$$

Once we have calculated the moments of the true distribution, we can calculate the parameters above, which fully determine the approximating distribution. The next step is to calculate the moments $\alpha_m(G)$ from (5.33) and then the cumulants $\kappa_m(G)$, which are obtained in a manner similar to (5.27)-(5.30).

The density function $g(x)$ of the lognormal distribution with parameters ν and λ^2 is

$$g(x) = \frac{1}{\lambda\sqrt{2\pi}}\frac{1}{x}\exp(\frac{-(\nu - \ln x)^2}{2\lambda^2}), \quad x > 0 \tag{5.38}$$

From (5.7) it seems as if we need to find the first four derivatives of this function, but it turns out that we only need two of them. The first derivative of g with respect to x is

$$\frac{dg}{dx}(x) = g(x)\frac{1}{x}(\frac{\nu - \ln x}{\lambda^2} - 1), \quad x > 0 \tag{5.39}$$

and the second derivative is

$$\frac{d^2g}{dx^2}(x) = g(x)\frac{1}{x^2}((\frac{\nu - \ln x}{\lambda^2} - 1)(\frac{\nu - \ln x}{\lambda^2} - 2) - \frac{1}{\lambda^2}), \quad x > 0 \qquad (5.40)$$

Once again the details appear from Appendix B.

5.2.4 Option price

We can rewrite (5.11), so the option price is

$$\begin{aligned} C &= \exp(-rT)E[(A-K)^+] \\ &= \exp(-rT)\frac{S_0}{n}E[(Y-k)^+] \\ &= \exp(-rT)\frac{S_0}{n}\int_k^\infty (x-k)f(x)dx \qquad (5.41) \end{aligned}$$

where f is the true density function of Y.

Lemma 7 *Applying the Edgeworth series expansion (5.7) to the integral in (5.41) leads to*

$$\begin{aligned} \int_k^\infty (x-k)f(x)dx &= \int_k^\infty (x-k)g(x)dx \\ &+\frac{\kappa_2(F)-\kappa_2(G)}{2!}g(k) - \frac{\kappa_3(F)-\kappa_3(G)}{3!}\frac{dg}{dx}(k) \\ &+\frac{\kappa_4(F)-\kappa_4(G)+3(\kappa_2(F)-\kappa_2(G))^2}{4!}\frac{d^2g}{dx^2}(k) \\ &+\varepsilon(k) \qquad (5.42) \end{aligned}$$

where G is the approximating lognormal distribution.

Proof. Substituting $f(x)$ from (5.7) for $f(x)$ in

$$\int_k^\infty (x-k)f(x)dx$$

gives integrals of the type

$$\int_k^\infty (x-k)\frac{d^m g}{dx^m}(x)dx, \quad m = 2, 3, 4$$

Using integration by parts, we get

$$\begin{aligned}
\int_k^\infty (x-k)\frac{d^m g}{dx^m}(x)dx &= \lim_{x\to\infty}\{(x-k)\frac{d^{m-1}g}{dx^{m-1}}(x)\} - (k-k)\frac{d^{m-1}g}{dx^{m-1}}(k) \\
&\quad - \int_k^\infty \frac{d^{m-1}g}{dx^{m-1}}(x)dx \\
&= \lim_{x\to\infty}\{x\frac{d^{m-1}g}{dx^{m-1}}(x)\} - k\lim_{x\to\infty}\frac{d^{m-1}g}{dx^{m-1}}(x) \\
&\quad -(\lim_{x\to\infty}\frac{d^{m-2}g}{dx^{m-2}}(x) - \frac{d^{m-2}g}{dx^{m-2}}(k)) \\
&= \frac{d^{m-2}g}{dx^{m-2}}(k)
\end{aligned}$$

where all the limits are zero, because the lognormal density $g(x)$ tends to zero in the following sense

$$\lim_{x\to\infty} x^u g(x) = 0, \quad \forall u > 0$$

which also makes the derivatives tend to zero. Now, (5.42) follows easily. ■

The next theorem tells us how to calculate the approximate option price.

Theorem 8 *The Turnbull and Wakeman price at time 0 of the Asian option with maturity T and strike K is*

$$\begin{aligned}
C &= \exp(-rT)\frac{S_0}{n}\{\exp(\nu+\frac{1}{2}\lambda^2)\Phi(d_1) - k\Phi(d_2) \\
&\quad + \frac{\kappa_2(F)-\kappa_2(G)}{2!}g(k) - \frac{\kappa_3(F)-\kappa_3(G)}{3!}\frac{dg}{dx}(k) \\
&\quad + \frac{\kappa_4(F)-\kappa_4(G)+3(\kappa_2(F)-\kappa_2(G))^2}{4!}\frac{d^2g}{dx^2}(k)\} \qquad (5.43)
\end{aligned}$$

where Φ is the standard normal distribution function and

$$\begin{aligned}
d_1 &= \frac{\nu+\lambda^2-\ln k}{\lambda} \qquad (5.44) \\
d_2 &= d_1 - \lambda \qquad (5.45)
\end{aligned}$$

Note that the term with the second cumulants in (5.43) equals zero in our setup (by choice of approximating distribution) and can therefore be removed from the expression.

Proof. As shown below

$$\int_k^\infty (x-k)g(x)dx = \exp(\nu + \frac{1}{2}\lambda^2)\Phi(\frac{\nu+\lambda^2-\ln k}{\lambda}) - k\Phi(\frac{\nu-\ln k}{\lambda})$$

Combining this with (5.41) and (5.42) gives the result in (5.43).
First, we split the integral into two integrals

$$\int_k^\infty (x-k)g(x)dx = \int_k^\infty xg(x)dx - k\int_k^\infty g(x)dx$$

Using the change of variables

$$y = \frac{\nu+\lambda^2-\ln x}{\lambda}, \quad \frac{dy}{dx} = -\frac{1}{\lambda x}$$

and the density (5.38), the first integral is

$$\begin{aligned}
\int_k^\infty xg(x)dx &= \int_k^\infty x\frac{1}{\lambda\sqrt{2\pi}}\frac{1}{x}\exp(\frac{-(\nu-\ln x)^2}{2\lambda^2})dx \\
&= \int_k^\infty x\frac{1}{\sqrt{2\pi}}\frac{1}{\lambda x}\exp(-\frac{1}{2}(\frac{\nu-\ln x}{\lambda})^2)dx \\
&= \int_{\frac{\nu+\lambda^2-\ln k}{\lambda}}^{-\infty} -\exp(\nu+\lambda^2-\lambda y)\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}(y-\lambda)^2)dy \\
&= \int_{-\infty}^{\frac{\nu+\lambda^2-\ln k}{\lambda}} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2 - \frac{1}{2}\lambda^2 + \lambda y + \nu + \lambda^2 - \lambda y)dy \\
&= \exp(\nu+\frac{1}{2}\lambda^2)\int_{-\infty}^{\frac{\nu+\lambda^2-\ln k}{\lambda}} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2)dy \\
&= \exp(\nu+\frac{1}{2}\lambda^2)\Phi(\frac{\nu+\lambda^2-\ln k}{\lambda})
\end{aligned}$$

Likewise, we make the change of variables

$$z = \frac{\nu-\ln x}{\lambda}, \quad \frac{dz}{dx} = -\frac{1}{\lambda x}$$

and the second integral is then

$$\begin{aligned}
\int_k^\infty g(x)dx &= \int_k^\infty \frac{1}{\lambda\sqrt{2\pi}}\frac{1}{x}\exp(\frac{-(\nu-\ln x)^2}{2\lambda^2})dx \\
&= \int_k^\infty \frac{1}{\sqrt{2\pi}}\frac{1}{\lambda x}\exp(-\frac{1}{2}(\frac{\nu-\ln x}{\lambda})^2)dx \\
&= \int_{\frac{\nu-\ln k}{\lambda}}^{-\infty} -\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)dz \\
&= \int_{-\infty}^{\frac{\nu-\ln k}{\lambda}} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)dz \\
&= \Phi(\frac{\nu-\ln k}{\lambda})
\end{aligned}$$

which completes the proof. ■

5.2.5 Method overview

The Turnbull and Wakeman algorithm for pricing Asian options can be summarized as follows

- Calculate the first four moments of the true distribution of the (scaled) average stock price.
- Match the first two moments of the approximating lognormal distribution to the true distribution.
- Calculate the third and fourth moments of the approximating distribution.
- Find the first and second derivatives of the approximating density function.
- Calculate the first four cumulants of both distributions.
- Calculate the Asian option price using the generalized Edgeworth series expansion.

5.3 Numerical results

Table 5.1 contains option prices for different values of stock price volatility σ, strike price K, number of prices in the average n and maturity T. The Monte

Carlo simulation has been carried out with 50,000 paths and the geometric option price as control variate, cf. Chapter 4. Unless otherwise mentioned the stock dividend, the interest rate and the initial stock price have been fixed at

$$q = 0, \quad r = 0.05, \quad S_0 = 100 \tag{5.46}$$

The Turnbull and Wakeman price is within 1 cent of the Monte Carlo estimate for all values of K and n and for small values of σ and T. The accuracy deteriorates for volatilities larger than 30% p.a. and especially for maturities longer than $1\frac{1}{2}$years.

We stated in Section 5.1 that nothing can be said about the error term $\varepsilon(x)$ in general. In the special case where all the moments exist (as in our setup), we have

$$\varepsilon_N(x) \to 0 \text{ uniformly as } N \to \infty \tag{5.47}$$

where N represents the number of terms included in the series expansion (see [Jarrow and Rudd, 1982]). This is very reassuring but for all practical purposes N is finite, so how does it influence the accuracy if we add or remove correction terms? To investigate this we have included the components of the pricing expression (5.43) in Table 5.2. The notation is

$$\begin{aligned}
term2 &= -\exp(-rT)\frac{S_0}{n}\frac{\kappa_3(F)-\kappa_3(G)}{3!}\frac{dg}{dx}(k), \\
term3 &= \exp(-rT)\frac{S_0}{n}\frac{\kappa_4(F)-\kappa_4(G)+3(\kappa_2(F)-\kappa_2(G))^2}{4!}\frac{d^2g}{dx^2}(k), \\
dg &= \frac{dg}{dx}(k), \qquad d^2g = \frac{d^2g}{dx^2}(k), \qquad c3/6 = \frac{\kappa_3(F)-\kappa_3(G)}{3!}, \\
c4/24 &= \frac{\kappa_4(F)-\kappa_4(G)+3(\kappa_2(F)-\kappa_2(G))^2}{4!}
\end{aligned} \tag{5.48}$$

Note that the correction term $term2$ is small (less than 3 cents) except for large volatilities whereas $term3$ plays an important role for large values of both σ and T. This is a consequence of a considerable difference in kurtosis between the true distribution and the approximating lognormal distribution as seen in the column labelled $c4/24$.

The last two columns of Table 5.1 show the Turnbull and Wakeman price with no correction for differences in fourth cumulants (TW3c) and with no corrections at all (Levy). The last column is labelled Levy because this pricing method is equivalent to the one used in [Levy, 1992]. He also refers to it as the "Wilkinson approximation" (setting the first two moments equal).

Figure 5.1 and Figure 5.2 display these prices less the Monte Carlo estimate for different values of σ, T and K.

Figure 5.1 shows that the Turnbull and Wakeman algorithm performs better than the other two and that not adjusting for cumulant differences (Levy) is better than only adjusting for third cumulants (TW3c). It also appears from Figure 5.1 that the TW algorithm is very sensitive to increases in maturity and should definitely not be used for maturities longer than 3 years. For $T > 3.5$ both Levy and TW3c perform better (but not well!) which is due to the large downward kurtosis correction $term3$, cf. Table 5.2.

Figure 5.2 is very interesting. We see that the TW price is within 1 cent of the MC estimate for all strike prices whereas TW3c varies up to 2 cents and Levy varies up to 3 cents. It depends on K which method is the better one, and hence we should not use (5.47) in combination with intuition to say that adding more correction terms gives better accuracy. When the option is deep-in-the-money TW overprices it and then underprices it as K is getting closer to S_0. This changes again when the option is out-of-the-money. To some extent, TW3c behaves in the opposite way while Levy is too large for in-the-money and at-the-money options and too small for out-of-the-money options. The explanations of these fluctuations can be found in Table 5.2 in the columns labelled dg and d^2g. The variations of the first and second order derivatives of the approximating density function cause the correction terms $term2$ and $term3$ to change sign and magnitude. Figure 5.2 shows that $term2$ is negative when $K < S_0$ and positive when $K > S_0$. $term3$ is negative when K is near S_0 and positive elsewhere.

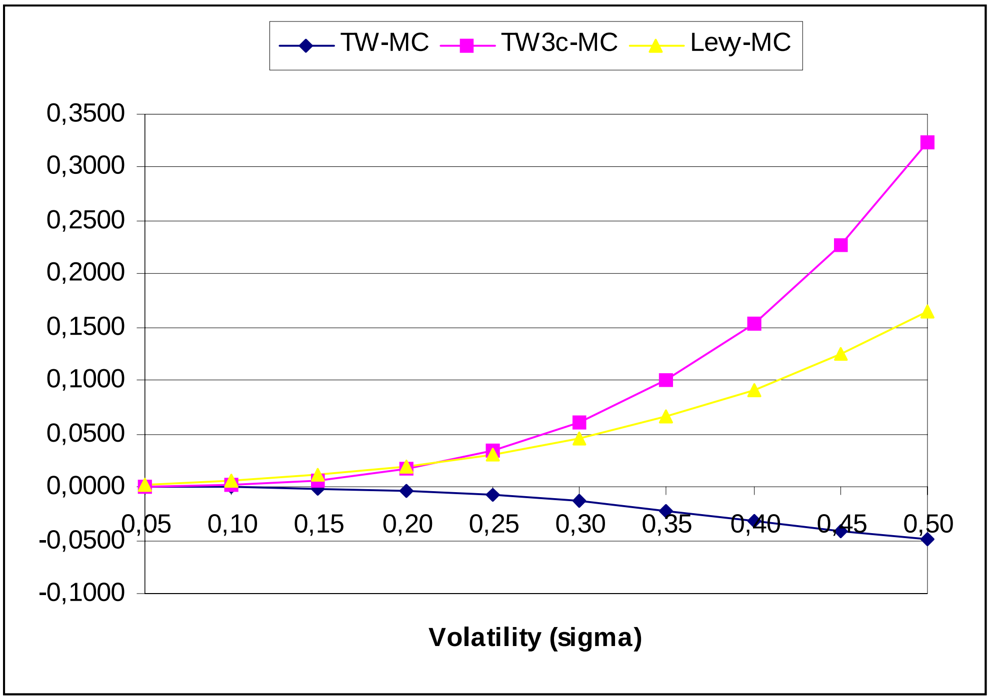
Finally, we note from Table 5.1 that increasing the number of prices in the average n does not affect the accuracy of the Turnbull and Wakeman algorithm, but the option prices decrease towards the continuous-time-averaging prices.

σ	K	n	T	MC	sd	TW	TW3c	Levy
0.05	100	12	1	2.93	0.0001	2.93	2.93	2.93
0.10	100	12	1	3.90	0.0004	3.90	3.91	3.91
0.15	100	12	1	5.01	0.0009	5.01	5.02	5.02
0.20	100	12	1	6.16	0.0016	6.15	6.17	6.17
0.25	100	12	1	7.31	0.0024	7.30	7.35	7.34
0.30	100	12	1	8.47	0.0035	8.46	8.53	8.52
0.35	100	12	1	9.64	0.0049	9.61	9.74	9.70
0.40	100	12	1	10.80	0.0065	10.77	10.95	10.89
0.45	100	12	1	11.96	0.0085	11.92	12.18	12.08
0.50	100	12	1	13.11	0.0107	13.07	13.44	13.28
0.20	100	12	**0.5**	4.11	0.0007	4.11	4.12	4.12
0.20	100	12	**1.0**	6.16	0.0016	6.15	6.17	6.17
0.20	100	12	**1.5**	7.84	0.0025	7.83	7.87	7.87
0.20	100	12	**2.0**	9.32	0.0034	9.30	9.36	9.37
0.20	100	12	**2.5**	10.66	0.0045	10.62	10.72	10.73
0.20	100	12	**3.0**	11.90	0.0055	11.83	11.98	11.99
0.20	100	12	**3.5**	13.05	0.0066	12.95	13.15	13.16
0.20	100	12	**4.0**	14.14	0.0078	13.99	14.25	14.26
0.20	100	12	**4.5**	15.16	0.0090	14.96	15.29	15.30
0.20	100	12	**5.0**	16.12	0.0102	15.86	16.27	16.29
0.20	100	**12**	1	6.16	0.0016	6.15	6.17	6.17
0.20	100	**24**	1	5.96	0.0016	5.96	5.98	5.98
0.20	100	**36**	1	5.90	0.0016	5.89	5.91	5.91
0.20	100	**48**	1	5.86	0.0016	5.86	5.88	5.88
0.20	100	**60**	1	5.84	0.0016	5.84	5.86	5.86
0.20	**70**	12	1	31.16	0.0015	31.16	31.16	31.16
0.20	**75**	12	1	26.42	0.0015	26.42	26.41	26.42
0.20	**80**	12	1	21.72	0.0015	21.72	21.71	21.73
0.20	**85**	12	1	17.16	0.0015	17.16	17.15	17.19
0.20	**90**	12	1	12.92	0.0015	12.91	12.92	12.95
0.20	**95**	12	1	9.19	0.0015	9.18	9.20	9.22
0.20	**100**	12	1	6.16	0.0016	6.15	6.17	6.17
0.20	**105**	12	1	3.87	0.0016	3.87	3.89	3.87
0.20	**110**	12	1	2.29	0.0015	2.30	2.30	2.28
0.20	**115**	12	1	1.28	0.0015	1.28	1.28	1.25
0.20	**120**	12	1	0.67	0.0013	0.68	0.67	0.65
0.20	**125**	12	1	0.34	0.0012	0.34	0.33	0.32
0.20	**130**	12	1	0.16	0.0010	0.16	0.16	0.15

Table 5.1: Asian option prices for different values of volatility, strike price, number of averaging times and maturity. MC: Monte Carlo simulation. sd: Standard deviation of estimator. TW: Turnbull and Wakeman price. (Continued in Table 5.2).

σ	K	n	T	term2	term3	c3/6	dg	c4/24	d^2g
0.05	100	12	1	0.00	0.00	0.0001	1.6724	0.0000	-1.7742
0.10	100	12	1	0.00	0.00	0.0021	0.2330	0.0003	-0.8139
0.15	100	12	1	0.00	-0.01	0.0109	0.0485	0.0037	-0.2795
0.20	100	12	1	0.00	-0.02	0.0353	0.0062	0.0216	-0.1214
0.25	100	12	1	0.00	-0.04	0.0888	-0.0064	0.0865	-0.0615
0.30	100	12	1	0.02	-0.07	0.1912	-0.0106	0.2739	-0.0345
0.35	100	12	1	0.03	-0.12	0.3702	-0.0118	0.7403	-0.0208
0.40	100	12	1	0.06	-0.19	0.6647	-0.0118	1.7883	-0.0131
0.45	100	12	1	0.10	-0.27	1.1286	-0.0114	3.9766	-0.0085
0.50	100	12	1	0.16	-0.37	1.8367	-0.0108	8.3069	-0.0056
0.20	100	12	**0.5**	0.00	-0.01	0.0082	0.0092	0.0024	-0.3485
0.20	100	12	**1.0**	0.00	-0.02	0.0353	0.0062	0.0216	-0.1214
0.20	100	12	**1.5**	0.00	-0.04	0.0856	0.0049	0.0813	-0.0651
0.20	100	12	**2.0**	-0.01	-0.07	0.1639	0.0041	0.2150	-0.0416
0.20	100	12	**2.5**	-0.01	-0.10	0.2761	0.0035	0.4685	-0.0293
0.20	100	12	**3.0**	-0.01	-0.14	0.4287	0.0030	0.9038	-0.0220
0.20	100	12	**3.5**	-0.01	-0.19	0.6295	0.0027	1.6032	-0.0172
0.20	100	12	**4.0**	-0.01	-0.25	0.8871	0.0024	2.6749	-0.0139
0.20	100	12	**4.5**	-0.02	-0.32	1.2118	0.0022	4.2597	-0.0115
0.20	100	12	**5.0**	-0.02	-0.41	1.6152	0.0020	6.5391	-0.0096
0.20	100	**12**	1	0.00	-0.02	0.0353	0.0062	0.0216	-0.1214
0.20	100	**24**	1	0.00	-0.02	0.2826	0.0018	0.3265	-0.0166
0.20	100	**36**	1	0.00	-0.02	0.9529	0.0008	1.6194	-0.0051
0.20	100	**48**	1	0.00	-0.02	2.2576	0.0005	5.0648	-0.0022
0.20	100	**60**	1	0.00	-0.02	4.4076	0.0003	12.2871	-0.0011
0.20	**70**	12	1	0.00	0.00	0.0353	0.0105	0.0216	0.0248
0.20	**75**	12	1	-0.01	0.01	0.0353	0.0346	0.0216	0.0566
0.20	**80**	12	1	-0.02	0.01	0.0353	0.0758	0.0216	0.0751
0.20	**85**	12	1	-0.03	0.01	0.0353	0.1145	0.0216	0.0442
0.20	**90**	12	1	-0.03	-0.01	0.0353	0.1193	0.0216	-0.0324
0.20	**95**	12	1	-0.02	-0.02	0.0353	0.0768	0.0216	-0.1034
0.20	**100**	12	1	0.00	-0.02	0.0353	0.0062	0.0216	-0.1214
0.20	**105**	12	1	0.02	-0.01	0.0353	-0.0577	0.0216	-0.0850
0.20	**110**	12	1	0.03	0.00	0.0353	-0.0916	0.0216	-0.0279
0.20	**115**	12	1	0.03	0.00	0.0353	-0.0938	0.0216	0.0170
0.20	**120**	12	1	0.02	0.01	0.0353	-0.0763	0.0216	0.0373
0.20	**125**	12	1	0.01	0.01	0.0353	-0.0529	0.0216	0.0380
0.20	**130**	12	1	0.01	0.01	0.0353	-0.0325	0.0216	0.0293

Table 5.2: (Continued from Table 5.1). Components of the Turnbull and Wakeman pricing expression (5.43). See (5.48) for an explanation of the column headers.



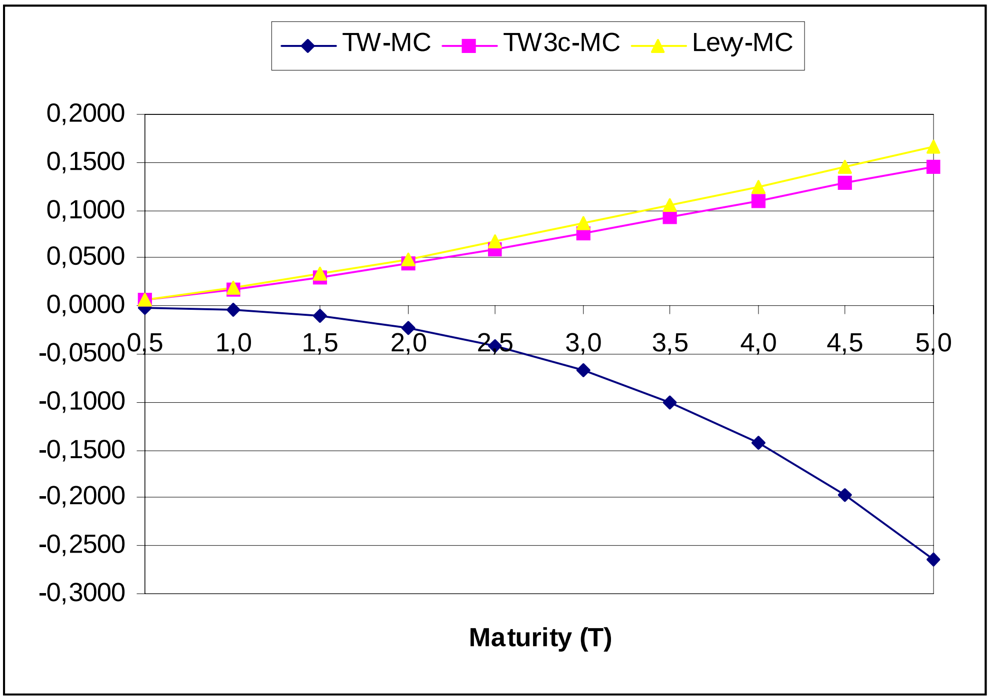
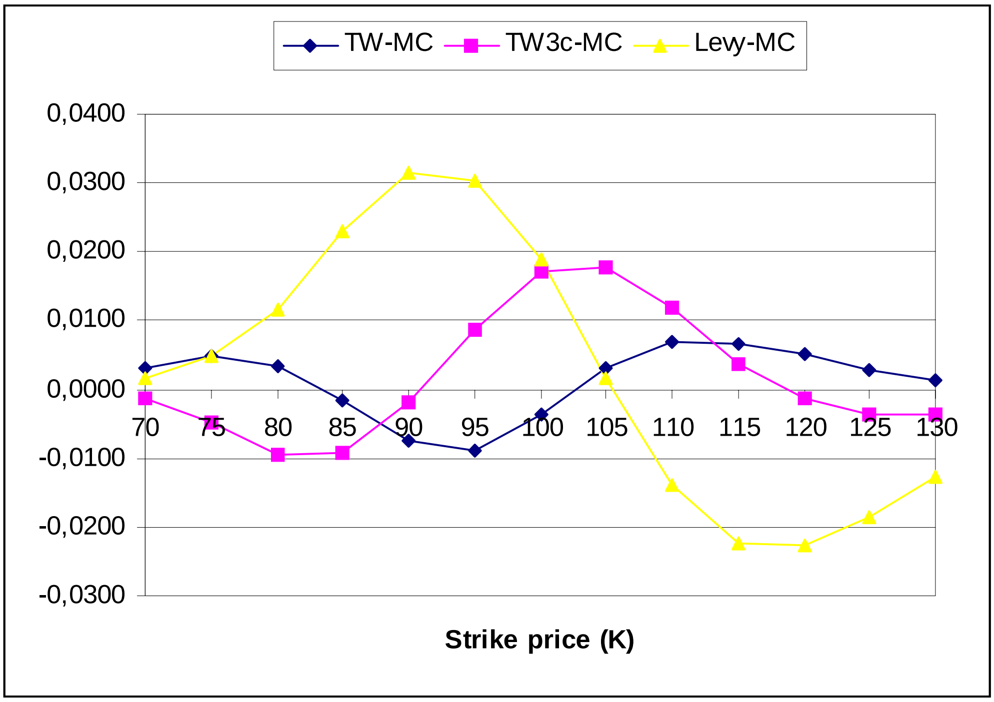


Figure 5.1: Option prices subtracted by Monte Carlo estimate. TW: Turnbull and Wakeman price. TW3c: TW with correction for differences in third cumulants only. Levy: TW without correction for differences in cumulants.



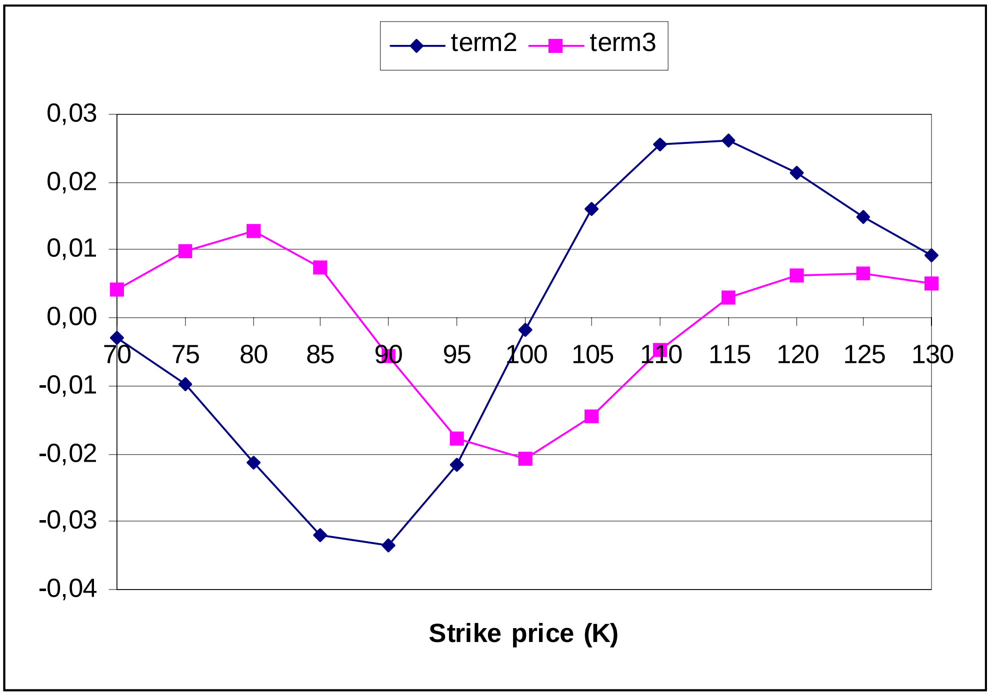


Figure 5.2: Top: Option prices subtracted by Monte Carlo estimate. Bottom: Contribution of correction terms to Turnbull and Wakeman price. See (5.48).

5.3.1 Conclusion

We have seen that the accuracy of the Turnbull and Wakeman algorithm is good when the stock price volatility is low and when the maturity of the option is short. The algorithm should be used with caution when the volatility exceeds 20-30% p.a. and it should never be used to price Asian options with a maturity longer than 2-3 years. The generalized Edgeworth series expansion adjusts for differences in kurtosis (and skewness), which results in huge pricing errors for options with long maturities. A possible way to fix this is to set the fourth moments equal instead of the second moments (5.31), but that is outside the scope of this thesis. Finally, we conclude that adding more terms to the Edgeworth series expansion does not necessarily improve the accuracy of the approximation.

Chapter 6

Milevsky and Posner method

[Milevsky and Posner, 1998] are inspired by previous attempts to find a good approximation for the distribution of the arithmetic average and use this to price Asian options. Some of the most frequently quoted articles in this context are [Levy, 1992], [Vorst, 1992] and [Turnbull and Wakeman, 1991]. Milevsky and Posner argue that the reciprocal gamma distribution should be preferred to the lognormal distribution as approximating distribution. This is due to the fact that the infinite sum of correlated lognormal random variables is reciprocally gamma distributed. The resulting pricing formula bears a close resemblance to the Black-Scholes formula and the method is very fast and easy to implement.

6.1 Reciprocal gamma distribution

6.1.1 Gamma distribution

Let $X \sim \Gamma(\alpha, \beta)$ denote that the random variable X is gamma distributed with parameters $\alpha > 0$ and $\beta > 0$. The density function g of X is

$$g(x) = \Gamma(\alpha)^{-1}\beta^{-\alpha}x^{\alpha-1}\exp(-\beta^{-1}x), \quad x > 0 \tag{6.1}$$

where Γ is the gamma function (see [Hoffmann-Jørgensen, 1994a] p. 216). To emphasize what the parameters are, we shall also use the notation $g(x|\alpha, \beta)$ and likewise for the distribution function $G(x|\alpha, \beta)$. Some useful properties are collected in the following proposition.

Proposition 9 *Let* $\alpha > 0$ *and* $\beta > 0$*. Then*

$$G(x|\alpha,\beta) = G(\frac{x}{\beta}|\alpha,1), \quad \forall x > 0 \tag{6.2}$$

$$g(x|\alpha,\beta) = \frac{1}{\beta}g(\frac{x}{\beta}|\alpha,1), \quad \forall x > 0 \tag{6.3}$$

$$g(x|\alpha,\beta) = \frac{x}{\beta(\alpha-1)}g(x|\alpha-1,\beta), \quad \forall x > 0, \quad \forall \alpha > 1 \tag{6.4}$$

Proof. See Appendix C. ∎

6.1.2 Reciprocal gamma distribution

Define $Y = \frac{1}{X}$. The random variable Y is reciprocally gamma distributed with parameters α and β. The distribution function G_R of Y is related to the distribution function G of X in the following way

$$\begin{aligned} G_R(y) &= P(Y \le y) = P(\frac{1}{Y} \ge \frac{1}{y}) = P(X \ge \frac{1}{y}) = 1 - P(X \le \frac{1}{y}) \\ &= 1 - G(\frac{1}{y}), \quad y > 0 \end{aligned} \tag{6.5}$$

Since the gamma distribution is absolutely continuous, $\frac{d}{dy}(G(y)) = g(y)$, and therefore the density function g_R of the reciprocal gamma distribution is

$$\begin{aligned} g_R(y) &= \frac{d}{dy}(G_R(y)) = \frac{d}{dy}(1 - G(\frac{1}{y})) = 0 - \frac{-1}{y^2}g(\frac{1}{y}) \\ &= \frac{1}{y^2}g(\frac{1}{y}), \quad y > 0 \end{aligned} \tag{6.6}$$

It follows that all results with the reciprocal gamma distribution can be rewritten and expressed by means of the gamma distribution instead. This is convenient since the gamma distribution is included in many software packages like MS Excel etc.

As shown in Appendix C, the moments M_i of the reciprocal gamma distribution are

$$M_i = E[Y^i] = E[\frac{1}{X^i}] = \frac{1}{\beta^i(\alpha-1)(\alpha-2)...(\alpha-i)}, \quad i \in \mathbb{N}, \quad i < \alpha \tag{6.7}$$

and the parameters α and β can be calculated from the first two moments which, therefore, fully specify the distribution

$$\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2} \tag{6.8}$$

$$\beta = \frac{M_2 - M_1^2}{M_1 M_2} \tag{6.9}$$

6.2 Method

6.2.1 Model

The stock price is given by

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \tag{6.10}$$

and the arithmetic average is

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{6.11}$$

where the averaging dates are

$$t_i = ih, \quad h = \frac{T}{n}, \quad i = 1, 2, ..., n \tag{6.12}$$

For continuous averaging, we have

$$A = \frac{1}{T}\int_0^T S_u du \tag{6.13}$$

The option payoff at maturity T is $\max\{A - K, 0\}$, so the option price at time 0 is

$$\begin{aligned} C &= \exp(-rT)E[(A-K)^+] \\ &= \exp(-rT)\int_{-\infty}^{\infty}(y-K)^+ f(y)dy \\ &= \exp(-rT)\int_{K}^{\infty}(y-K)f(y)dy \end{aligned} \tag{6.14}$$

where f is the true, but unknown, density function of A. We approximate f with the density function g_R of a reciprocal gamma distribution. Fortunately we can calculate the moments of the true distribution, so we can match these with the moments of the approximating distribution in order to get a good fit. The approximate price is then

$$C = \exp(-rT)\int_K^{\infty}(y-K)g_R(y)dy \tag{6.15}$$

6.2.2 Discrete averaging

Define

$$I = \frac{1}{S_0}\sum_{i=1}^{n} S_{t_i} \tag{6.16}$$

which is the sum of the averaging date prices scaled by the current price. Later in this chapter, we shall see that the limiting distribution (as $T \to \infty$) of the continuous averaging version of I is a reciprocal gamma distribution. This is the theoretical justification for approximating the average with the reciprocal gamma distribution. The relationship between I and the arithmetic average A is

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} = \frac{S_0}{n} I \tag{6.17}$$

The first two risk neutral moments of the average are for $r \neq q$

$$E[A] = \frac{S_0}{n} e^{(r-q)h}\left(\frac{1 - e^{(r-q)nh}}{1 - e^{(r-q)h}}\right) \tag{6.18}$$

$$E[A^2] = \frac{S_0^2}{n^2}(a_1 - a_2 + a_3 - a_4) \tag{6.19}$$

where

$$a_1 = \frac{e^{(2(r-q)+\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q)h})(1 - e^{(2(r-q)+\sigma^2)h})} \tag{6.20}$$

$$a_2 = \frac{e^{((r-q)(n+2)+\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h})} \tag{6.21}$$

$$a_3 = \frac{e^{(3(r-q)+\sigma^2)h} - e^{((r-q)(n+2)+\sigma^2)h}}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h})} \tag{6.22}$$

$$a_4 = \frac{e^{(4(r-q)+2\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q+\sigma^2)h})(1 - e^{(2(r-q)+\sigma^2)h})} \tag{6.23}$$

(6.18)-(6.19) are modified versions of the formulae in [Levy, 1992] (see Appendix C for details). We choose the parameters of the approximating reciprocal gamma distribution so its moments are equal to the above moments of the true distribution. From (6.8) and (6.9), we get

$$\alpha = \frac{2E[A^2] - E[A]^2}{E[A^2] - E[A]^2} \tag{6.24}$$

$$\beta = \frac{E[A^2] - E[A]^2}{E[A]E[A^2]} \tag{6.25}$$

We can now calculate the option price from the following theorem.

Theorem 10 *The Milevsky and Posner price at time 0 of the Asian option with maturity T and strike K is*

$$C = \exp(-rT)E[A]G(\frac{1}{K}|\alpha-1,\beta) - \exp(-rT)KG(\frac{1}{K}|\alpha,\beta) \tag{6.26}$$

where $G(\cdot|\alpha,\beta)$ is the distribution function of the gamma distribution with parameters α and β obtained from (6.24) and (6.25).

Proof. From (6.15) and (6.6)

$$\begin{aligned} C &= \exp(-rT)\int_K^\infty (y-K)g_R(y|\alpha,\beta)dy \\ &= \exp(-rT)\int_K^\infty (y-K)\frac{1}{y^2}g(\frac{1}{y}|\alpha,\beta)dy \end{aligned}$$

We change variables $x = \frac{1}{y}$ so $\frac{dx}{dy} = -\frac{1}{y^2}$, which leads to

$$\begin{aligned} C &= \exp(-rT)\int_{\frac{1}{K}}^0 -(\frac{1}{x}-K)g(x|\alpha,\beta)dx \\ &= \exp(-rT)\int_0^{\frac{1}{K}} (\frac{1}{x}-K)g(x|\alpha,\beta)dx \\ &= \exp(-rT)\int_0^{\frac{1}{K}} \frac{1}{x}g(x|\alpha,\beta)dx - \exp(-rT)K\int_0^{\frac{1}{K}} g(x|\alpha,\beta)dx \\ &= \exp(-rT)\int_0^{\frac{1}{K}} \frac{1}{\beta(\alpha-1)}g(x|\alpha-1,\beta)dx - \exp(-rT)KG(\frac{1}{K}|\alpha,\beta) \\ &= \exp(-rT)\frac{1}{\beta(\alpha-1)}G(\frac{1}{K}|\alpha-1,\beta) - \exp(-rT)KG(\frac{1}{K}|\alpha,\beta) \end{aligned}$$

where we have used (6.4). Since $M_1 = E[A]$ by choice of approximating distribution and $M_1 = \frac{1}{\beta(\alpha-1)}$ from (6.7), we get the wanted expression

$$C = \exp(-rT)E[A]G(\frac{1}{K}|\alpha-1,\beta) - \exp(-rT)KG(\frac{1}{K}|\alpha,\beta)$$

■

Note how the pricing formula (6.26) is similar to the Black-Scholes formula (2.3) with $\Phi(d_1)$ and $\Phi(d_2)$ replaced with $G(\frac{1}{K}|\alpha-1,\beta)$ and $G(\frac{1}{K}|\alpha,\beta)$.

6.2.3 Method overview

The Milevsky and Posner algorithm can be summarized as follows

- Calculate the first two moments of the arithmetic average of the stock price.
- Match the first two moments of the approximating distribution to the moments above and calculate the parameters of the reciprocal gamma distribution.
- Calculate the Asian option price using Theorem 10.

6.2.4 Continuous averaging

For the sake of completeness, we shall also take a look at the Milevsky and Posner method with continuous averaging. Define the scaled integral

$$I = \frac{1}{S_0} \int_0^T S_u du \tag{6.27}$$

so the continuous average is

$$A = \frac{1}{T} \int_0^T S_u du = \frac{S_0}{T} I \tag{6.28}$$

The limiting distribution of I is the reciprocal gamma distribution as stated in the next theorem.

Theorem 11 *Let*

$$I_\infty = \lim_{T \to \infty} I = \frac{1}{S_0} \lim_{T \to \infty} \int_0^T S_u du \tag{6.29}$$

Assume $r - q - \frac{1}{2}\sigma^2 < 0$. *Then* I_∞ *is reciprocally gamma distributed with parameters* $1 + \frac{2(q-r)}{\sigma^2}$ *and* $\frac{1}{2}\sigma^2$ *which is equivalent to*

$$I_\infty^{-1} \sim \Gamma(1 + \frac{2(q-r)}{\sigma^2}, \frac{1}{2}\sigma^2) \tag{6.30}$$

Proof. See Appendix A of [Milevsky and Posner, 1998] ■

Note that the result is only valid when $r - q - \frac{1}{2}\sigma^2 < 0$. This means that we should expect the Milevsky and Posner method to perform best when the volatility σ is high and/or when the stock dividend q is greater than the interest rate r.

According to [Milevsky and Posner, 1998], the first two risk neutral moments of the average are for $r \neq q$

$$E[A] = S_0 \frac{\exp((r-q)T) - 1}{(r-q)T} \tag{6.31}$$

$$\begin{aligned} E[A^2] &= 2\frac{S_0^2}{T^2}\{\frac{\exp((2(r-q)+\sigma^2)T}{(r-q+\sigma^2)(2r-2q+\sigma^2)} \\ &+ \frac{1}{r-q}[\frac{1}{2(r-q)+\sigma^2} - \frac{\exp((r-q)T)}{r-q+\sigma^2}]\} \end{aligned} \tag{6.32}$$

and for $r = q$

$$E[A] = S_0 \tag{6.33}$$

$$E[A^2] = 2\frac{S_0^2}{T^2}\frac{\exp(\sigma^2 T) - 1 - \sigma^2 T}{\sigma^4} \tag{6.34}$$

We can now calculate the parameters α and β of the approximating reciprocal gamma distribution from (6.24) and (6.25). The pricing formula is given in the following theorem.

Theorem 12 *The Milevsky and Posner price at time 0 of the continuous averaging Asian option with maturity T and strike K is*

$$\begin{aligned} C &= S_0 \frac{\exp(-qT) - \exp(-rT)}{(r-q)T} G(\frac{1}{K}|\alpha - 1, \beta) \\ &- \exp(-rT) K G(\frac{1}{K}|\alpha, \beta) \end{aligned} \tag{6.35}$$

for $r \neq q$ and

$$C = S_0 \exp(-rT) G(\frac{1}{K}|\alpha - 1, \beta) - \exp(-rT) K G(\frac{1}{K}|\alpha, \beta) \tag{6.36}$$

for $r = q$.

Proof. Insert (6.31) and (6.33) respectively into (6.26). ∎

6.3 Numerical results

Some of the questions we would like to answer in this section are

1. Is the reciprocal gamma distribution a better choice for approximating the arithmetic average than the lognormal distribution? In other words, is the Milevsky and Posner (MP) method better than the Levy method?

2. How is the accuracy of the MP method affected when we increase the stock dividend q?

3. Can we improve the MP method by correcting for differences in skewness and kurtosis as the Turnbull and Wakeman (TW) method does?

4. What is the relationship between the discrete and continuous averaging prices?

Table 6.1 contains option prices for different values of stock price volatility σ, strike price K, number of prices in the average n and maturity T. The Monte Carlo simulation has been carried out with 50,000 paths and the geometric option price as control variate, cf. Chapter 4. Unless otherwise mentioned the stock dividend, the interest rate and the initial stock price have been fixed at

$$q = 0, \quad r = 0.05, \quad S_0 = 100 \tag{6.37}$$

We have included both discrete and continuous averaging prices, cf. (6.26) and (6.35), as well as prices obtained by using a generalized Edgeworth series expansion (MP2). The reciprocal gamma distribution has all the necessary properties to allow us to develop similar results as in Chapter 5. Thus, from a theoretical viewpoint MP2 has the potential to be just as good as the Turnbull and Wakeman method. Unfortunately this is not the case as we shall see.

To answer the first question we look at Figure 6.1 and Figure 6.2. It appears that the accuracy of the Milevsky and Posner method is more or less similar to the accuracy of the Levy method, cf. Chapter 5. The important difference is that when MP underprices then Levy overprices and vice versa. Based on this, an obvious way to improve the accuracy is to use the average of the two prices. Numerical results (illustrated in Section 7.2 and Section 8.2) show that this approach is better than any of the methods we have seen so far, but also that it can not compete with the Curran method in Chapter 8. Since the MP and Levy methods match the approximating distribution to the first two moments of the true distribution, the price differences observed are caused by the higher moments. Figure 6.2 shows the density of the lognormal distribution subtracted by the density of the reciprocal gamma distribution.

The assumption in Theorem 11 was that $r - q - \frac{1}{2}\sigma^2 < 0$, but Figure 6.1 shows that the MP method is not good for pricing options with high volatilities. Let us then take a look at the stock dividend q. Table 6.2 contains prices of Asian options with various strike prices for three different values of the stock dividend. These prices are illustrated in Figure 6.3 where we see that an increase in q leads to a parallel shift along the first axis. The magnitude of the maximum deviation from the Monte Carlo estimate is approximately the same for different values of q, so we conclude that the MP method is not more precise for high stock dividends.

We saw in Chapter 5 that the TW algorithm performed better than the Levy method in most cases. This has inspired us to improve the MP method in a similar fashion. The prices labelled MP2 have been calculated by means of the Turnbull and Wakeman algorithm with a reciprocal gamma distribution as approximating distribution. The moments and cumulants are obtained from (6.24)-(6.25), (6.7) and (5.27)-(5.30) and the derivatives of the reciprocal gamma density appear from Appendix C. Figure 6.1 and Figure 6.2 show that the accuracy has improved for all volatilities and for most of the strike prices. In contrast, it is evident from Figure 6.1 that MP2 is extremely sensitive to increases in maturity and performs worse than the MP method for $T > 2$. Moreover, comparisons (see Section 7.2) show that the TW prices in general are better than the MP2 prices.

Table 6.3 and Figure 6.3 show how the discrete averaging prices converge to the continuous averaging prices for both MP and MP2. Even for 120 prices in the average there is a difference of approximately 4-6 cents, which implies that it is not recommendable to use models with continuous averaging when the option in question is based on a discrete average.

σ	K	n	T	MC	sd	MP	MP c	MP2
0.05	100	12	1	2.93	0.0001	2.93	2.71	n/a
0.10	100	12	1	3.90	0.0004	3.90	3.64	n/a
0.15	100	12	1	5.01	0.0009	5.00	4.68	n/a
0.20	100	12	1	6.16	0.0016	6.13	5.75	6.16
0.25	100	12	1	7.31	0.0024	7.27	6.83	7.33
0.30	100	12	1	8.47	0.0035	8.42	7.91	8.50
0.35	100	12	1	9.64	0.0049	9.55	8.98	9.68
0.40	100	12	1	10.80	0.0065	10.68	10.05	10.86
0.45	100	12	1	11.96	0.0085	11.80	11.12	12.04
0.50	100	12	1	13.11	0.0107	12.90	12.17	13.21
0.20	100	12	**0.5**	4.11	0.0007	4.10	3.85	n/a
0.20	100	12	**1.0**	6.16	0.0016	6.13	5.75	6.16
0.20	100	12	**1.5**	7.84	0.0025	7.80	7.30	7.86
0.20	100	12	**2.0**	9.32	0.0034	9.26	8.66	9.37
0.20	100	12	**2.5**	10.66	0.0045	10.58	9.89	10.75
0.20	100	12	**3.0**	11.90	0.0055	11.79	11.02	12.06
0.20	100	12	**3.5**	13.05	0.0066	12.92	12.07	13.30
0.20	100	12	**4.0**	14.14	0.0078	13.98	13.05	14.51
0.20	100	12	**4.5**	15.16	0.0090	14.97	13.97	15.70
0.20	100	12	**5.0**	16.12	0.0102	15.91	14.84	16.89
0.20	100	**12**	1	6.16	0.0016	6.13	5.75	6.16
0.20	100	**24**	1	5.96	0.0016	5.94	5.75	5.97
0.20	100	**36**	1	5.90	0.0016	5.88	5.75	5.90
0.20	100	**48**	1	5.86	0.0016	5.84	5.75	5.87
0.20	100	**60**	1	5.84	0.0016	5.82	5.75	5.85
0.20	**70**	12	1	31.16	0.0015	31.16	30.96	31.16
0.20	**75**	12	1	26.42	0.0015	26.41	26.20	26.41
0.20	**80**	12	1	21.72	0.0015	21.71	21.49	21.71
0.20	**85**	12	1	17.16	0.0015	17.14	16.88	17.17
0.20	**90**	12	1	12.92	0.0015	12.88	12.57	12.94
0.20	**95**	12	1	9.19	0.0015	9.16	8.79	9.21
0.20	**100**	12	1	6.16	0.0016	6.13	5.75	6.16
0.20	**105**	12	1	3.87	0.0016	3.87	3.51	3.87
0.20	**110**	12	1	2.29	0.0015	2.31	2.00	2.28
0.20	**115**	12	1	1.28	0.0015	1.30	1.07	1.27
0.20	**120**	12	1	0.67	0.0013	0.70	0.55	0.67
0.20	**125**	12	1	0.34	0.0012	0.36	0.26	0.34
0.20	**130**	12	1	0.16	0.0010	0.18	0.12	0.16

Table 6.1: Option prices for different volatilities, maturities, strike prices and number of prices in the average. MP: Milevsky and Posner. MP c: MP with continuous averaging. MP2: MP with cumulant correction à la Turnbull and Wakeman.

σ	**K**	**n**	**q**	**T**	**MC**	**sd**	**MP**	**MP c**	**MP2**
0.20	**70**	12	**0.04**	1	29.06	0.0013	29.06	29.02	29.05
0.20	**75**	12	0.04	1	24.32	0.0013	24.32	24.27	24.31
0.20	**80**	12	0.04	1	19.66	0.0013	19.64	19.57	19.65
0.20	**85**	12	0.04	1	15.19	0.0013	15.16	15.04	15.20
0.20	**90**	12	0.04	1	11.11	0.0013	11.08	10.88	11.13
0.20	**95**	12	0.04	1	7.65	0.0013	7.62	7.36	7.66
0.20	**100**	12	0.04	1	4.93	0.0013	4.92	4.63	4.93
0.20	**105**	12	0.04	1	2.97	0.0013	2.98	2.71	2.97
0.20	**110**	12	0.04	1	1.68	0.0013	1.71	1.48	1.67
0.20	**115**	12	0.04	1	0.90	0.0012	0.92	0.76	0.89
0.20	**120**	12	0.04	1	0.45	0.0011	0.48	0.37	0.45
0.20	**125**	12	0.04	1	0.22	0.0009	0.24	0.17	0.22
0.20	**130**	12	0.04	1	0.10	0.0007	0.11	0.08	0.10
0.20	**70**	12	**0.08**	1	27.01	0.0012	27.01	27.13	27.01
0.20	**75**	12	0.08	1	22.29	0.0012	22.28	22.39	22.28
0.20	**80**	12	0.08	1	17.67	0.0012	17.65	17.72	17.67
0.20	**85**	12	0.08	1	13.31	0.0012	13.28	13.28	13.33
0.20	**90**	12	0.08	1	9.44	0.0012	9.41	9.32	9.46
0.20	**95**	12	0.08	1	6.27	0.0012	6.24	6.07	6.27
0.20	**100**	12	0.08	1	3.88	0.0011	3.87	3.66	3.87
0.20	**105**	12	0.08	1	2.24	0.0011	2.25	2.05	2.23
0.20	**110**	12	0.08	1	1.21	0.0011	1.24	1.07	1.20
0.20	**115**	12	0.08	1	0.62	0.0010	0.64	0.53	0.61
0.20	**120**	12	0.08	1	0.30	0.0008	0.32	0.24	0.30
0.20	**125**	12	0.08	1	0.14	0.0007	0.15	0.11	0.14
0.20	**130**	12	0.08	1	0.06	0.0006	0.07	0.05	0.06
0.20	**70**	12	**0.12**	1	25.03	0.0012	25.02	25.29	25.02
0.20	**75**	12	0.12	1	20.32	0.0012	20.31	20.56	20.32
0.20	**80**	12	0.12	1	15.77	0.0011	15.74	15.94	15.77
0.20	**85**	12	0.12	1	11.55	0.0011	11.52	11.62	11.57
0.20	**90**	12	0.12	1	7.92	0.0011	7.89	7.88	7.94
0.20	**95**	12	0.12	1	5.05	0.0010	5.04	4.93	5.06
0.20	**100**	12	0.12	1	3.00	0.0010	3.00	2.85	2.99
0.20	**105**	12	0.12	1	1.65	0.0009	1.67	1.53	1.64
0.20	**110**	12	0.12	1	0.85	0.0009	0.88	0.76	0.85
0.20	**115**	12	0.12	1	0.41	0.0008	0.44	0.36	0.41
0.20	**120**	12	0.12	1	0.19	0.0007	0.21	0.16	0.19
0.20	**125**	12	0.12	1	0.08	0.0005	0.10	0.07	0.09
0.20	**130**	12	0.12	1	0.03	0.0004	0.04	0.03	0.04

Table 6.2: Option prices for different strike prices and stock dividends. MP: Milevsky and Posner. MP c: MP with continuous averaging. MP2: MP with cumulant correction à la Turnbull and Wakeman.

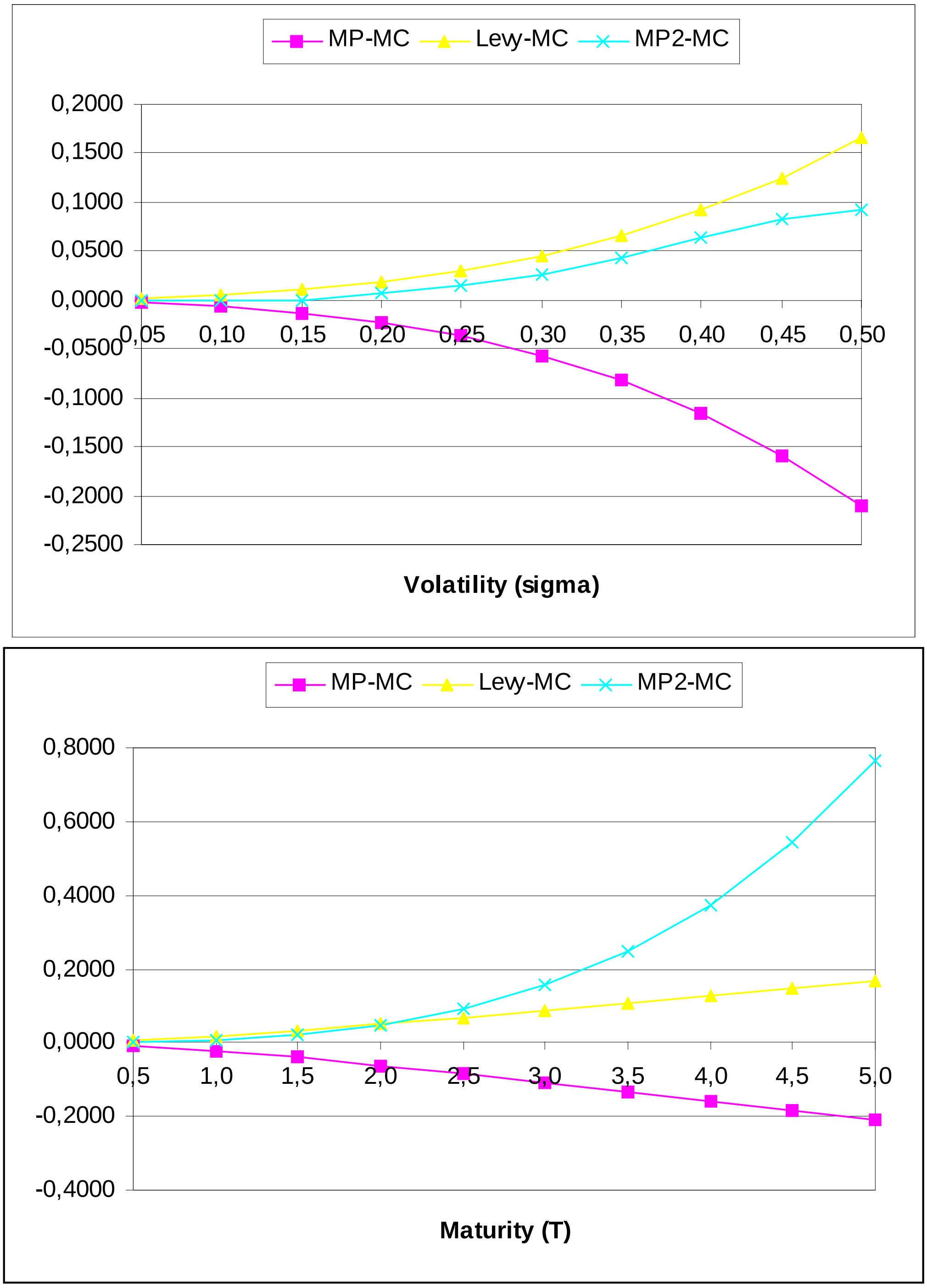


Figure 6.1: Option prices less Monte Carlo estimates. MP: Milevsky and Posner. MP2: MP with cumulant correction à la Turnbull and Wakeman.

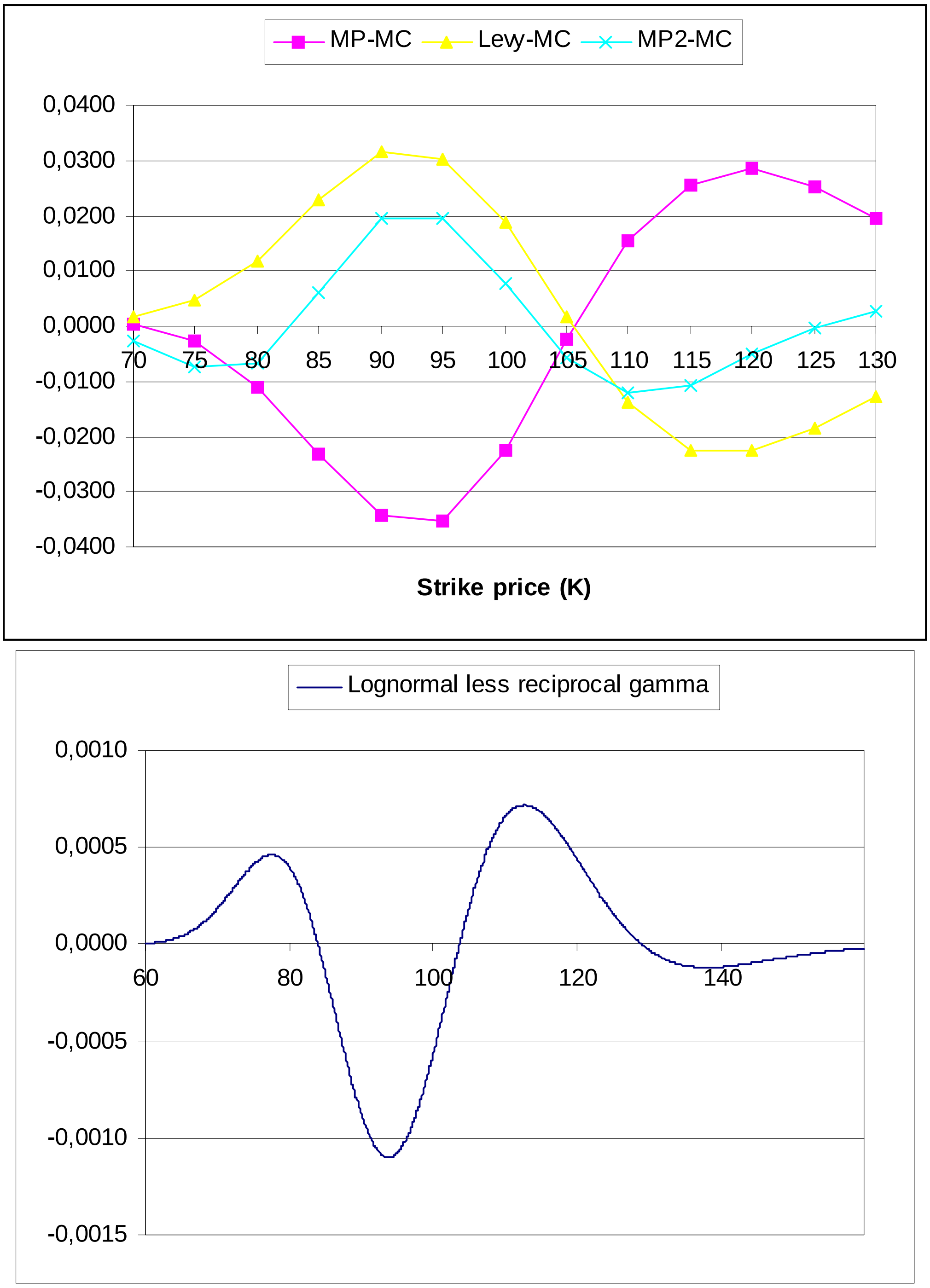
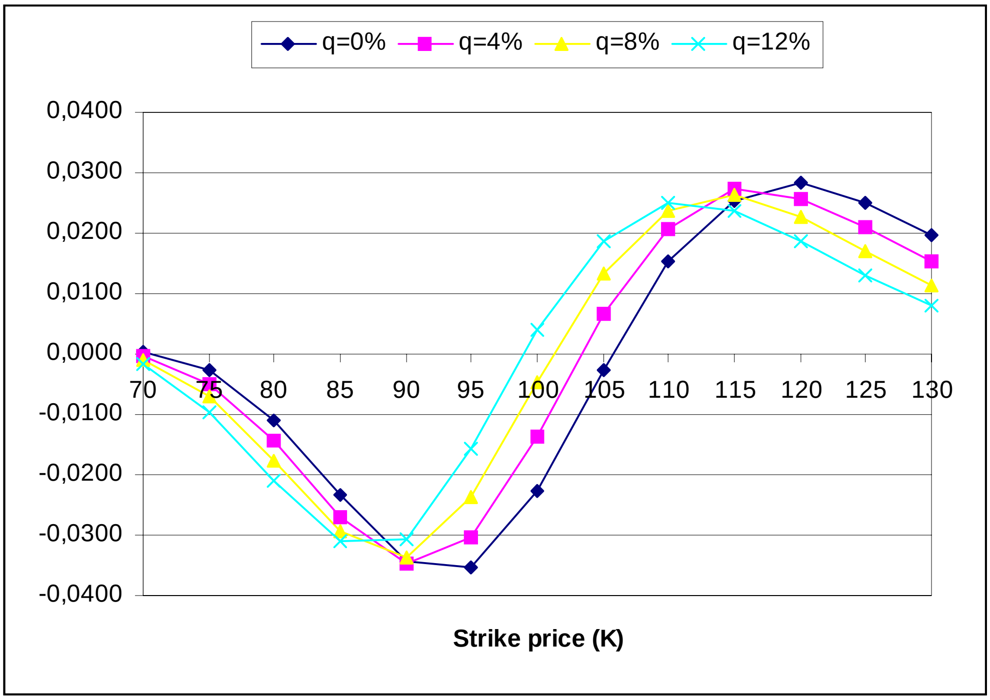


Figure 6.2: Top: Option prices less Monte Carlo estimates. MP: Milevsky and Posner. MP2: MP with cumulant correction à la Turnbull and Wakeman. Bottom: Difference between the lognormal density and the reciprocal gamma density for the same mean and variance.



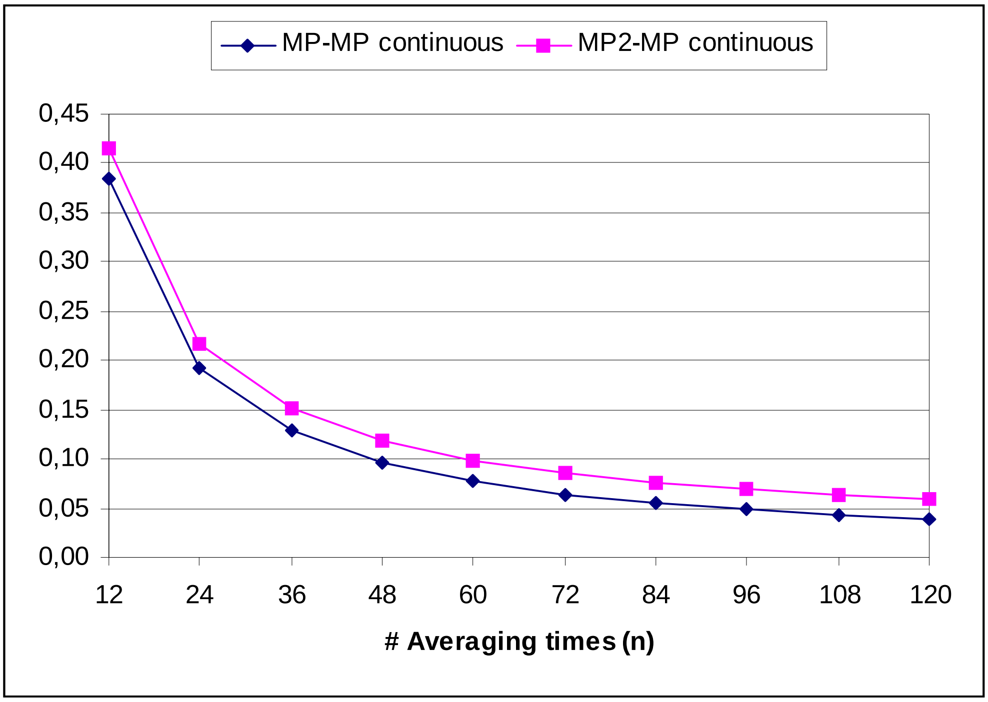


Figure 6.3: Top: Milevsky and Posner option prices less Monte Carlo estimates for different stock dividends q. Bottom: Difference between Milevsky and Posner option prices with discrete and continuous averaging. MP: Milevsky and Posner. MP2: MP with cumulant correction à la Turnbull and Wakeman.

σ	K	n	T	MP	MP c	MP2
0.20	100	**12**	1	6.13	5.75	6.16
0.20	100	**24**	1	5.94	5.75	5.97
0.20	100	**36**	1	5.88	5.75	5.90
0.20	100	**48**	1	5.84	5.75	5.87
0.20	100	**60**	1	5.82	5.75	5.85
0.20	100	**72**	1	5.81	5.75	5.83
0.20	100	**84**	1	5.80	5.75	5.82
0.20	100	**96**	1	5.80	5.75	5.82
0.20	100	**108**	1	5.79	5.75	5.81
0.20	100	**120**	1	5.79	5.75	5.81

Table 6.3: Option prices for different numbers of prices in the average. MP: Milevsky and Posner. MP c: MP with continuous averaging. MP2: MP with cumulant correction à la Turnbull and Wakeman.

6.3.1 Conclusion

Our results show no indication that the reciprocal gamma distribution is a better approximation to the distribution of the arithmetic stock price average than the lognormal distribution. The pricing errors of the Milevsky and Posner method and the Levy method seem to be negatively correlated, so we suggested that taking the average of the prices would improve the accuracy, which was supported by numerical results.

[Milevsky and Posner, 1998] state that the pricing method should perform at its best when $q > r$, but our results show no noteworthy increase in accuracy for options with high stock dividends. Increasing q for fixed K implies that the option's probability of finishing in-the-money is decreasing, which explains the shift seen in Figure 6.3.

We improved the Milevsky and Posner method to take differences in skewness and kurtosis into account but, as in the Turnbull and Wakeman case, the pricing errors got out of control for long maturities. Furthermore, this approach was dominated by the TW algorithm for most values of the strike price.

Finally, we concluded that even though the discrete average converge to the continuous average, there are substantial price differences for small numbers of prices in the average. Since all traded Asian options are based on a discrete average, this implies that continuous averaging models should be used with caution.

Chapter 7

Vorst method

[Vorst, 1992] gives an exact pricing formula for Asian options based on a geometric average. He uses this to approximate the arithmetic Asian option price by adjusting the strike price with the difference in expectation of the arithmetic and geometric averages. Furthermore, he obtains upper and lower bounds on the price.

7.1 Method

7.1.1 Model

The stock price is given by

$$S_t = S_0 \exp((r - q - \frac{1}{2}\sigma^2)t + \sigma W_t) \tag{7.1}$$

The geometric average is

$$G = (\prod_{i=1}^{n} S_{t_i})^{\frac{1}{n}} \tag{7.2}$$

and the arithmetic average is

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{7.3}$$

where the averaging times as usual are equidistant

$$t_i = ih, \quad h = \frac{T}{n}, \quad i = 1, 2, ..., n \tag{7.4}$$

7.1.2 Geometric Asian option

From (7.2) the natural logarithm of the geometric average is

$$\ln G = \frac{1}{n}\sum_{i=1}^{n}\ln S_{t_i} = \ln S_0 + \frac{1}{n}\sum_{i=1}^{n}\ln\frac{S_{t_i}}{S_0} \tag{7.5}$$

(7.1) gives that

$$\begin{aligned}\ln\frac{S_{t_i}}{S_0} &= (r-q-\frac{1}{2}\sigma^2)t_i + \sigma W_{t_i} \\ &\sim N((r-q-\frac{1}{2}\sigma^2)t_i, \sigma^2 t_i), \quad i = 1,2,...,n\end{aligned} \tag{7.6}$$

Since the Brownian Motion has independent increments, we conclude from (7.5) and (7.6) that $\ln G$ is normally distributed and we shall now find the mean μ_G and the variance σ_G^2.

The mean of $\ln G$ is

$$\begin{aligned}\mu_G &= \ln S_0 + \frac{1}{n}\sum_{i=1}^{n}E[\ln\frac{S_{t_i}}{S_0}] \\ &= \ln S_0 + \frac{1}{n}\sum_{i=1}^{n}(r-q-\frac{1}{2}\sigma^2)t_i \\ &= \ln S_0 + \frac{1}{n}(r-q-\frac{1}{2}\sigma^2)\sum_{i=1}^{n}ih \\ &= \ln S_0 + \frac{1}{n}(r-q-\frac{1}{2}\sigma^2)h\frac{n+1}{2}n \\ &= \ln S_0 + (r-q-\frac{1}{2}\sigma^2)\frac{T+h}{2}\end{aligned} \tag{7.7}$$

where we have used the summation formula

$$\sum_{i=1}^{n} i = \frac{n+1}{2}n \tag{7.8}$$

The variance of $\ln G$ is

$$\begin{aligned}
\sigma_G^2 &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} Cov(\ln\frac{S_{t_i}}{S_0}, \ln\frac{S_{t_j}}{S_0}) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma^2 Cov(W_{t_i}, W_{t_j}) \\
&= \frac{\sigma^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \min(t_i, t_j) \\
&= \frac{\sigma^2 h}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \min(i, j) \qquad (7.9)
\end{aligned}$$

To calculate the double sum we use the summation formula (7.8) and

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6} \qquad (7.10)$$

We get

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\min(i,j) &= \sum_{i=1}^{n}\sum_{j=1}^{i}\min(i,j) + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\min(i,j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i} j + \sum_{i=1}^{n}\sum_{j=i+1}^{n} i \\
&= \sum_{i=1}^{n}\frac{i+1}{2}i + \sum_{i=1}^{n}(n-i)i \\
&= \frac{1}{2}\sum_{i=1}^{n}(i^2 + i + 2ni - 2i^2) \\
&= \frac{1}{2}\sum_{i=1}^{n}((2n+1)i - i^2) \\
&= \frac{2n+1}{2}\sum_{i=1}^{n} i - \frac{1}{2}\sum_{i=1}^{n} i^2 \\
&= \frac{2n+1}{2}\frac{n+1}{2}n - \frac{1}{2}\frac{(2n+1)(n+1)n}{6} \\
&= \frac{(2n+1)(n+1)n}{6} \qquad (7.11)
\end{aligned}$$

The variance can now be expressed as

$$\begin{aligned}\sigma_G^2 &= \frac{\sigma^2 h}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\min(i,j) \\ &= \frac{\sigma^2 h}{n^2}\frac{(2n+1)(n+1)n}{6} \\ &= \sigma^2 h\frac{(2n+1)(n+1)}{6n} \qquad (7.12)\end{aligned}$$

For easier comparison with [Vorst, 1992], we can also write

$$\begin{aligned}\sigma_G^2 &= \sigma^2 h\frac{(2n-1)(n-1)+6n}{6n} \\ &= \sigma^2(h(n-1)\frac{(2n-1)}{6n}+h) \\ &= \sigma^2(h+(T-h)\frac{(2n-1)}{6n}) \qquad (7.13)\end{aligned}$$

We have seen that $\ln G \sim N(\mu_G, \sigma_G^2)$, so the geometric average G is lognormally distributed and we are now able to price the option based on the geometric average.

Theorem 13 *The price at time 0 of the geometric Asian option with strike K and maturity T is*

$$C_G = \exp(-rT)\{\exp(\mu_G + \frac{1}{2}\sigma_G^2)\Phi(d_1) - K\Phi(d_2)\} \qquad (7.14)$$

where

$$\begin{aligned}d_1 &= \frac{\mu_G - \ln K + \sigma_G^2}{\sigma_G} \qquad (7.15)\\ d_2 &= d_1 - \sigma_G \qquad (7.16)\end{aligned}$$

Proof. The payoff of the option is $Max(G-K,0)$ so the price is

$$\begin{aligned}C_G &= \exp(-rT)E[(G-K)^+] \\ &= \exp(-rT)\int_K^\infty (x-K)g(x)dx\end{aligned}$$

where g is the lognormal density function of G. The result now follows from the proof of Theorem 8. ■

7.1.3 Arithmetic option price and bounds

Since the geometric average is less than the arithmetic average, $G \leq A$, the geometric option price is a lower bound for the arithmetic option price

$$C_G = \exp(-rT)E[(G-K)^+] \leq \exp(-rT)E[(A-K)^+] = C \qquad (7.17)$$

The inequality

$$\begin{aligned} \max(A-K,0) &= \max(G-K, G-A) + A - G \\ &\leq \max(G-K,0) + A - G \end{aligned} \qquad (7.18)$$

leads to the upper bound

$$\begin{aligned} & \exp(-rT)E[\max(G-K,0) + A - G] \\ = & \exp(-rT)\{E[(G-K)^+] + E[A] - E[G]\} \\ = & C_G + \exp(-rT)(E[A] - E[G]) \end{aligned} \qquad (7.19)$$

Thus, we have obtained the following bounds on the arithmetic option price

$$C_G \leq C \leq C_G + \exp(-rT)(E[A] - E[G]) \qquad (7.20)$$

To calculate the upper bound we must calculate the mean of the arithmetic average. From (7.1), (7.3), (7.6) and (5.33)

$$\begin{aligned} E[A] &= E[\frac{1}{n}\sum_{i=1}^{n} S_{t_i}] \\ &= \frac{S_0}{n}\sum_{i=1}^{n} E[\frac{S_{t_i}}{S_0}] \\ &= \frac{S_0}{n}\sum_{i=1}^{n} \exp((r-q-\frac{1}{2}\sigma^2)t_i + \frac{1}{2}\sigma^2 t_i) \\ &= \frac{S_0}{n}\sum_{i=1}^{n} \exp((r-q)ih) \\ &= \frac{S_0}{n}\frac{e^{(r-q)h} - e^{(r-q)(n+1)h}}{1-e^{(r-q)h}} \\ &= \frac{S_0}{n}e^{(r-q)h}\frac{1-e^{(r-q)nh}}{1-e^{(r-q)h}} \end{aligned} \qquad (7.21)$$

where we have used the summation formula

$$\sum_{i=1}^{n} e^{ai} = \frac{e^a - e^{a(n+1)}}{1-e^a}, \quad a \neq 0 \qquad (7.22)$$

Since G is lognormally distributed with parameters μ_G and σ_G^2 the mean of the geometric average is

$$E[G] = \exp(\mu_G + \frac{1}{2}\sigma_G^2) \tag{7.23}$$

Once it has been established that a price is in a certain interval, it is tempting to approximate the price with the value of the interval midpoint. [Vorst, 1992] takes another approach since he approximates the arithmetic option price with the price of a geometric option with a downward adjusted strike. The rationale is that the geometric and arithmetic averages are strongly correlated and so are the prices of the corresponding options. Moreover, pricing with a lower strike will to some extent compensate for the fact that the geometric option price is always lower than the arithmetic one.

Theorem 14 *Define the Vorst price at time 0 of the arithmetic Asian option with strike K and maturity T as*

$$\tilde{C} = \exp(-rT)E[(G - K')^+] \tag{7.24}$$

where the adjusted strike price is

$$K' = K - (E[A] - E[G]) \tag{7.25}$$

The pricing formula is then

$$\tilde{C} = \exp(-rT)\{\exp(\mu_G + \frac{1}{2}\sigma_G^2)\Phi(d_1') - K'\Phi(d_2')\} \tag{7.26}$$

where μ_G and σ_G^2 are obtained from (7.7) and (7.12) and

$$d_1' = \frac{\mu_G - \ln K' + \sigma_G^2}{\sigma_G} \tag{7.27}$$

$$d_2' = d_1' - \sigma_G \tag{7.28}$$

The approximate option price $\tilde{C}$ is also in between the upper and lower bounds in (7.20), which implies that the pricing error is bounded by

$$\left|\tilde{C} - C\right| \leq \exp(-rT)(E[A] - E[G]) \tag{7.29}$$

Proof. (7.26) follows directly from applying Theorem 13 to the geometric Asian option with strike K', cf. (7.25). We have to show that the price $\tilde{C}$ is in between the bounds in (7.20). Since

$$\begin{aligned}\max(G - K, 0) &\leq \max(G - K + E[A] - E[G], 0) \\ &= \max(G - K, E[G] - E[A]) + E[A] - E[G] \\ &\leq \max(G - K, 0) + E[A] - E[G]\end{aligned}$$

it follows that

$$\begin{aligned} C_G &= \exp(-rT)E[(G-K)^+] \\ &\leq \exp(-rT)E[(G-K+E[A]-E[G])^+] \\ &= \tilde{C} \\ &\leq \exp(-rT)E[(G-K)^+ + E[A]-E[G]] \\ &= C_G + \exp(-rT)(E[A]-E[G]) \end{aligned}$$

The same bounds apply for the true price C, cf. (7.20), so we conclude that the absolute difference between $\tilde{C}$ and C is less than the difference between the upper bound and the lower bound

$$\begin{aligned} \left|\tilde{C} - C\right| &\leq C_G + \exp(-rT)(E[A]-E[G]) - C_G \\ &= \exp(-rT)(E[A]-E[G]) \end{aligned}$$

■

7.1.4 Method overview

The Vorst method can be summarized as follows

- Calculate the difference of the mean of the arithmetic and geometric averages.
- Reduce the strike price by the difference above.
- Use Theorem 14 to calculate the price of the arithmetic Asian option.
- Obtain the pricing bounds in (7.20) on the arithmetic Asian option by calculating the price of the geometric Asian option in Theorem 13.

7.2 Numerical results

Table 7.1 contains option prices for different values of stock price volatility σ, strike price K, number of prices in the average n and maturity T. The Monte Carlo simulation has been carried out with 50,000 paths and the geometric option price as control variate, cf. Chapter 4. Unless otherwise mentioned the stock dividend, the interest rate and the initial stock price have been fixed at

$$q = 0, \quad r = 0.05, \quad S_0 = 100 \tag{7.30}$$

We see that the bounds labelled up and C_G are very poor in the sense that they are far away from the Monte Carlo estimate. Figure 7.1 shows that the accuracy of the Vorst method deteriorates as volatility and maturity increase. It appears from Figure 7.2 that the Vorst method slightly overprices in-the-money options while at-the-money and out-of-the-money options are heavily underpriced. When we think of it, the substantial pricing errors should not come as a big surprise, since we approximate the arithmetic average with a lognormal distribution and only correct for the mean.

The table and the figures in this section have also been used to illustrate two points from the previous chapter. The first point is that taking the average of the Levy price and the Milevsky and Posner price (MPLevy/MP+L) leads to a superior accuracy compared to all the other methods we have considered so far. The second point is that the generalized Edgeworth series expansion is better suited for a lognormal distribution (TW) than for a reciprocal gamma distribution (MP2).

7.2.1 Conclusion

The Vorst method is the first method in this thesis to establish bounds on the option price. We saw that the same bounds applied to both the true price and the approximation but unfortunately they did not constrain the prices enough to be useful. Neither were the Vorst prices very encouraging but all our efforts have not been in vain. We used the geometric average to price options based on an arithmetic average but, as we shall see in the next chapter, we far from used all its potential.

σ	K	n	T	MC	sd	Vorst	up	C_G	MP+L
0.05	100	12	1	2.93	0.0001	2.93	2.93	2.90	2.93
0.10	100	12	1	3.90	0.0004	3.90	3.93	3.84	3.90
0.15	100	12	1	5.01	0.0009	5.00	5.07	4.88	5.01
0.20	100	12	1	6.16	0.0016	6.13	6.27	5.94	6.15
0.25	100	12	1	7.31	0.0024	7.27	7.51	6.99	7.31
0.30	100	12	1	8.47	0.0035	8.40	8.76	8.02	8.47
0.35	100	12	1	9.64	0.0049	9.53	10.03	9.04	9.63
0.40	100	12	1	10.80	0.0065	10.66	11.32	10.03	10.79
0.45	100	12	1	11.96	0.0085	11.77	12.63	11.00	11.94
0.50	100	12	1	13.11	0.0107	12.86	13.95	11.94	13.09
0.20	100	12	**0.5**	4.11	0.0007	4.10	4.17	4.01	4.11
0.20	100	12	**1.0**	6.16	0.0016	6.13	6.27	5.94	6.15
0.20	100	12	**1.5**	7.84	0.0025	7.78	8.00	7.50	7.83
0.20	100	12	**2.0**	9.32	0.0034	9.24	9.52	8.85	9.31
0.20	100	12	**2.5**	10.66	0.0045	10.55	10.89	10.05	10.65
0.20	100	12	**3.0**	11.90	0.0055	11.76	12.16	11.15	11.89
0.20	100	12	**3.5**	13.05	0.0066	12.88	13.34	12.16	13.04
0.20	100	12	**4.0**	14.14	0.0078	13.93	14.45	13.09	14.12
0.20	100	12	**4.5**	15.16	0.0090	14.91	15.49	13.96	15.14
0.20	100	12	**5.0**	16.12	0.0102	15.85	16.47	14.77	16.10
0.20	100	**12**	1	6.16	0.0016	6.13	6.27	5.94	6.15
0.20	100	**24**	1	5.96	0.0016	5.93	6.08	5.74	5.96
0.20	100	**36**	1	5.90	0.0016	5.86	6.01	5.68	5.89
0.20	100	**48**	1	5.86	0.0016	5.83	5.98	5.64	5.86
0.20	100	**60**	1	5.84	0.0016	5.81	5.96	5.63	5.84
0.20	**70**	12	1	31.16	0.0015	31.16	31.16	30.83	31.16
0.20	**75**	12	1	26.42	0.0015	26.42	26.42	26.09	26.42
0.20	**80**	12	1	21.72	0.0015	21.72	21.73	21.40	21.72
0.20	**85**	12	1	17.16	0.0015	17.17	17.20	16.86	17.16
0.20	**90**	12	1	12.92	0.0015	12.92	12.98	12.64	12.92
0.20	**95**	12	1	9.19	0.0015	9.18	9.28	8.95	9.19
0.20	**100**	12	1	6.16	0.0016	6.13	6.27	5.94	6.15
0.20	**105**	12	1	3.87	0.0016	3.82	4.02	3.69	3.87
0.20	**110**	12	1	2.29	0.0015	2.23	2.48	2.14	2.29
0.20	**115**	12	1	1.28	0.0015	1.22	1.50	1.17	1.28
0.20	**120**	12	1	0.67	0.0013	0.63	0.93	0.60	0.68
0.20	**125**	12	1	0.34	0.0012	0.30	0.62	0.29	0.34
0.20	**130**	12	1	0.16	0.0010	0.14	0.47	0.13	0.16

Table 7.1: Option prices and bounds for different volatilities, maturities, strike prices and number of prices in the average. up: Vorst upper bound. C_G: Lower bound (geometric option price). MP+L: Average of Levy price and Milevsky and Posner price.

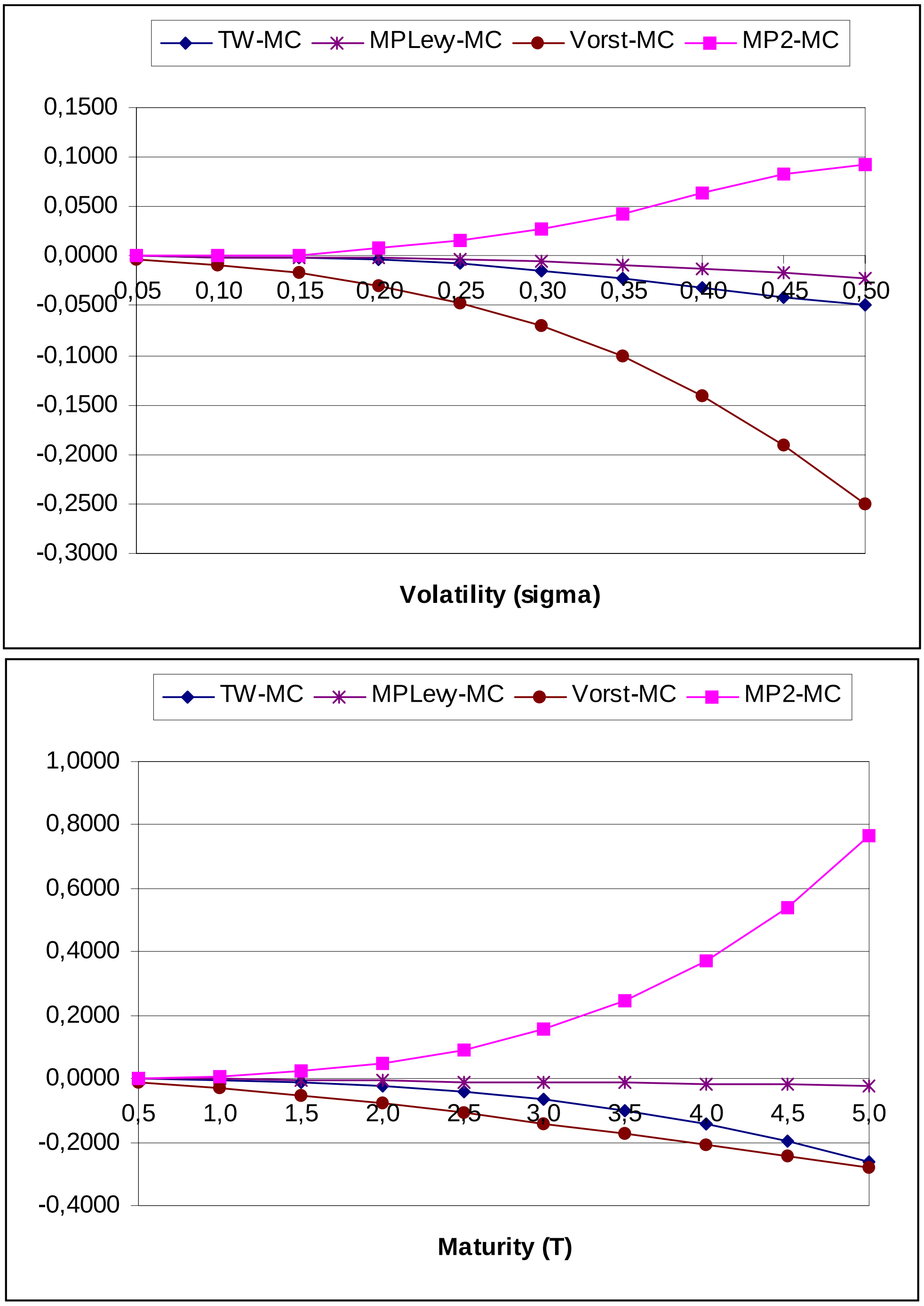


Figure 7.1: Vorst option prices less Monte Carlo estimates. TW, MPLevy and MP2 correspond to methods treated in previous chapters.

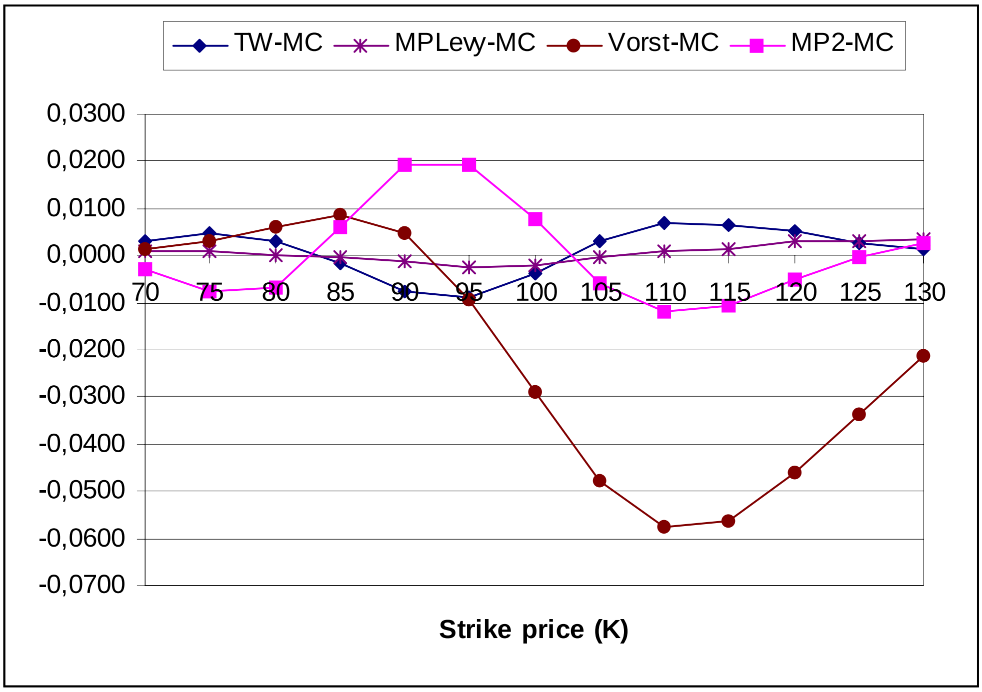


Figure 7.2: Vorst option prices less Monte Carlo estimates. TW, MPLevy and MP2 correspond to methods treated in previous chapters.

Chapter 8

Curran method

[Curran, 1994] introduces an extremely precise method for pricing Asian options and options on portfolios. The approach is to condition on the geometric average and integrate with respect to its lognormal distribution. Curran develops two different approximations, and we shall use the so-called naive approximation to obtain a lower bound on the Asian option price.

8.1 Method

8.1.1 Model

The stock price follows the stochastic differential equation

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \tag{8.1}$$

and the arithmetic average of the stock prices is

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{8.2}$$

where the time points are equidistant

$$t_i = ih, \quad h = \frac{T}{n}, \quad i = 1, 2, ..., n \tag{8.3}$$

(8.1) implies that the stock price at time t_i is lognormally distributed, so

$$\begin{aligned} \ln S_{t_i} &= \ln S_0 + (r - q - \frac{1}{2}\sigma^2)t_i + \sigma W_{t_i} \\ &\sim N(\mu_i, \sigma_i^2), \quad i = 1, 2, ..., n \end{aligned} \tag{8.4}$$

where

$$\mu_i = \ln S_0 + (r - q - \frac{1}{2}\sigma^2)t_i, \quad i = 1, 2, ..., n \tag{8.5}$$

$$\sigma_i^2 = \sigma^2 t_i, \quad i = 1, 2, ..., n \tag{8.6}$$

The geometric average is

$$G = (\prod_{i=1}^{n} S_{t_i})^{\frac{1}{n}} \tag{8.7}$$

and taking the natural logarithm leads to

$$\ln G = \frac{1}{n} \sum_{i=1}^{n} \ln S_{t_i} \tag{8.8}$$

From Chapter 7 we know that $\ln G$ is normally distributed with parameters μ_G and σ_G^2 given by

$$\mu_G = \ln S_0 + (r - q - \frac{1}{2}\sigma^2)\frac{T + h}{2} \tag{8.9}$$

$$\sigma_G^2 = \sigma^2 h \frac{(2n + 1)(n + 1)}{6n} \tag{8.10}$$

To sum up, the marginal distributions of $\ln S_{t_i}$ and $\ln G$ are normal, and it turns out that they also have a jointly normal distribution.

Proposition 15 *The random vector $\eta_i = (\ln S_{t_i}, \ln G)$ has a 2-dimensional normal distribution for $i = 1, 2, ..., n$ with mean vector*

$$(\mu_i, \mu_G), \quad i = 1, 2, ..., n \tag{8.11}$$

and covariance matrix

$$\begin{bmatrix} \sigma_i^2 & \gamma_i \\ \gamma_i & \sigma_G^2 \end{bmatrix}, \quad i = 1, 2, ..., n \tag{8.12}$$

where

$$\gamma_i = Cov(\ln S_{t_i}, \ln G) = \frac{\sigma^2 h}{2n}((2n + 1)i - i^2), \quad i = 1, 2, ..., n \tag{8.13}$$

Proof. Let $i \in \{1, 2, ..., n\}$. We can split the Wiener process at time t_n into independent increments

$$\begin{aligned} W_{t_n} &= (W_{t_n} - W_{t_{n-1}}) + (W_{t_{n-1}} - W_{t_{n-2}}) + ... + (W_{t_1} - W_0) + W_0 \\ &= \Delta W_{t_n} + \Delta W_{t_{n-1}} + ... + \Delta W_{t_1} \end{aligned}$$

Define the vector of increments

$$W^{\Delta} = (\Delta W_{t_1}, \Delta W_{t_2}, ..., \Delta W_{t_n})$$

which has a n-dimensional normal distribution. Since $\ln S_{t_i}$ and $\ln G$ are linear combinations of these increments, we can find a vector ξ_i and a matrix Σ_i, so

$$\eta_i = \xi_i + W^{\Delta}\Sigma_i$$

It follows from (4.22.16) in [Hoffmann-Jørgensen, 1994a] that the distribution of η_i is jointly normal. The covariance between $\ln S_{t_i}$ and $\ln G$ is

$$\begin{aligned}
\gamma_i &= Cov(\ln S_{t_i}, \ln G) = Cov(\ln S_{t_i}, \frac{1}{n}\sum_{j=1}^{n} \ln S_{t_j}) \\
&= \frac{1}{n}\sum_{j=1}^{n} Cov(\ln S_{t_i}, \ln S_{t_j}) = \frac{1}{n}\sum_{j=1}^{n} \sigma^2 \min(t_i, t_j) \\
&= \frac{\sigma^2}{n}\sum_{j=1}^{n} h \min(i,j) = \frac{\sigma^2 h}{n}(\sum_{j=1}^{i} \min(i,j) + \sum_{j=i+1}^{n} \min(i,j)) \\
&= \frac{\sigma^2 h}{n}(\sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i) = \frac{\sigma^2 h}{n}(\frac{i+1}{2} i + (n-i)i) \\
&= \frac{\sigma^2 h}{n}(ni + \frac{1}{2}i - \frac{1}{2}i^2) = \frac{\sigma^2 h}{2n}((2n+1)i - i^2)
\end{aligned}$$

■

The next proposition shows that the normal distribution is stable to conditioning when the random variables are jointly normal.

Proposition 16 *The conditional distribution of* $\ln S_{t_i}$ *given* $\ln G$ *is a normal distribution*

$$(\ln S_{t_i} | \ln G = y) \sim N(\mu_i + (y - \mu_G)\frac{\gamma_i}{\sigma_G^2}, \sigma_i^2 - \frac{\gamma_i^2}{\sigma_G^2}), \quad i = 1, 2, ..., n \quad (8.14)$$

Moreover, $(S_{t_i} | \ln G)$ *has a lognormal distribution and*

$$E[S_{t_i} | \ln G = y] = \exp(\mu_i + (y - \mu_G)\frac{\gamma_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\gamma_i^2}{\sigma_G^2})), \quad i = 1, 2, ..., n \quad (8.15)$$

Proof. We conclude from (10.7.6) in [Hoffmann-Jørgensen, 1994b] and Proposition 15 that $(\ln S_{t_i} | \ln G)$ is normally distributed. The calculation of the mean and the variance in (8.14) is straightforward, but has been left out. It follows per definition that $(S_{t_i} | \ln G)$ is lognormally distributed with the same parameters, and the conditional expectation (8.15) is then calculated from (5.33). ■

8.1.2 Bounds on the option price

We condition on the geometric average G in the pricing expression of the Asian option

$$\begin{aligned} C &= e^{-rT} E[(A-K)^+] \\ &= e^{-rT} E[E[(A-K)^+ | G]] \\ &= e^{-rT} \int_0^\infty E[(A-K)^+ | G = x] g(x) dx \end{aligned} \tag{8.16}$$

where g is the lognormal density function of G. Define

$$C_1 = \int_0^K E[(A-K)^+ | G = x] g(x) dx \tag{8.17}$$

$$C_2 = \int_K^\infty E[(A-K)^+ | G = x] g(x) dx \tag{8.18}$$

so

$$C = e^{-rT}(C_1 + C_2) \tag{8.19}$$

Since the arithmetic average is greater than the geometric average $A \geq G$

$$C_2 = \int_K^\infty E[A - K | G = x] g(x) dx \tag{8.20}$$

Loosely speaking, C_1 corresponds to the situation where the geometric option finishes out-of-the-money and the status of the arithmetic option is unknown. Likewise, C_2 corresponds to the situation where both options finish in-the-money. Proposition 16 enables us to calculate C_2 from (8.20), but we have to settle for an approximation of C_1. Using Jensen's inequality we obtain a lower bound on C_1

$$\begin{aligned} C_1 &= \int_0^K E[(A-K)^+|G=x]g(x)dx \\ &\geq \int_0^K (E[A-K|G=x])^+ g(x)dx \\ &= \int_L^K E[A-K|G=x]g(x)dx \\ &= \tilde{C}_1 \end{aligned} \tag{8.21}$$

where

$$L = \{x|E[A|G=x]=K\} \tag{8.22}$$

To find L we calculate the conditional expectation

$$\begin{aligned} E[A|G=x] = E[\frac{1}{n}\sum_{i=1}^n S_{t_i}|G=x] &= \frac{1}{n}\sum_{i=1}^n E[S_{t_i}|\ln G = \ln x] \\ &= \frac{1}{n}\sum_{i=1}^n \exp(\mu_i + (\ln x - \mu_G)\frac{\gamma_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\gamma_i^2}{\sigma_G^2})) \end{aligned} \tag{8.23}$$

where we have used (8.15). Setting the last expression equal to K and solving for x is difficult, so we have applied a simple numerical method, which appears from the Visual Basic code in Appendix E. It is important that the approximation $\tilde{L}$ is greater than L to make sure that $\tilde{C}_1$ is a lower bound on C_1.

Theorem 17 *A lower bound on the price of the Asian option with strike K and maturity T is*

$$\begin{aligned} \tilde{C} &= \exp(-rT)(\tilde{C}_1 + C_2) \\ &= \exp(-rT)\{\frac{1}{n}\sum_{i=1}^n \exp(\mu_i + \frac{1}{2}\sigma_i^2)\Phi(\frac{\mu_G - \ln L + \gamma_i}{\sigma_G}) \\ &\quad -K\Phi(\frac{\mu_G - \ln L}{\sigma_G})\} \end{aligned} \tag{8.24}$$

where the relevant parameters are given in (8.5), (8.6), (8.9), (8.10), (8.13) and (8.22).

Proof. Collecting (8.21) and (8.20) gives

$$\begin{aligned}
\tilde{C}_1 + C_2 &= \int_L^\infty E[A-K|G=x]g(x)dx \\
&= \int_L^\infty E[A|G=x]g(x)dx - K\int_L^\infty g(x)dx \\
&= \int_L^\infty E[\frac{1}{n}\sum_{i=1}^n S_{t_i}|G=x]g(x)dx - K\int_L^\infty g(x)dx \\
&= \int_L^\infty \frac{1}{n}\sum_{i=1}^n E[S_{t_i}|G=x]g(x)dx - K\int_L^\infty g(x)dx \\
&= \frac{1}{n}\sum_{i=1}^n \int_L^\infty E[S_{t_i}|\ln G=\ln x]g(x)dx - K\int_L^\infty g(x)dx
\end{aligned}$$

We know from the proof of Theorem 8 that

$$K\int_L^\infty g(x)dx = K\Phi(\frac{\mu_G - \ln L}{\sigma_G})$$

and from (8.15)

$$\begin{aligned}
&\int_L^\infty E[S_{t_i}|\ln G=\ln x]g(x)dx \\
&\quad = \int_L^\infty \exp(\mu_i + (\ln x - \mu_G)\frac{\gamma_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\gamma_i^2}{\sigma_G^2}))g(x)dx \\
&\quad = \exp(\mu_i + \frac{1}{2}\sigma_i^2)\int_L^\infty \exp((\ln x - \mu_G)\frac{\gamma_i}{\sigma_G^2} - \frac{1}{2}\frac{\gamma_i^2}{\sigma_G^2})g(x)dx
\end{aligned}$$

Using the density (5.38) of the lognormal distribution the last integral is

$$\int_L^\infty \frac{1}{\sqrt{2\pi}\sigma_G x}\exp((\ln x - \mu_G)\frac{\gamma_i}{\sigma_G^2} - \frac{1}{2}\frac{\gamma_i^2}{\sigma_G^2} - \frac{1}{2}(\frac{\mu_G - \ln x}{\sigma_G})^2)dx$$

We make the change of variables

$$z = \frac{\mu_G - \ln x + \gamma_i}{\sigma_G}, \quad \frac{dz}{dx} = -\frac{1}{\sigma_G x}$$

which leads to

$$\begin{aligned}
&\int_{\frac{\mu_G-\ln L+\gamma_i}{\sigma_G}}^{-\infty} -\frac{1}{\sqrt{2\pi}}\exp((\frac{\gamma_i}{\sigma_G}-z)\frac{\gamma_i}{\sigma_G}-\frac{1}{2}\frac{\gamma_i^2}{\sigma_G^2}-\frac{1}{2}(z-\frac{\gamma_i}{\sigma_G})^2)dz \\
= &\int_{-\infty}^{\frac{\mu_G-\ln L+\gamma_i}{\sigma_G}} \frac{1}{\sqrt{2\pi}}\exp(-z\frac{\gamma_i}{\sigma_G}+\frac{1}{2}\frac{\gamma_i^2}{\sigma_G^2}-\frac{1}{2}z^2-\frac{1}{2}\frac{\gamma_i^2}{\sigma_G^2}+z\frac{\gamma_i}{\sigma_G})dz \\
= &\int_{-\infty}^{\frac{\mu_G-\ln L+\gamma_i}{\sigma_G}} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)dz \\
= &\Phi(\frac{\mu_G-\ln L+\gamma_i}{\sigma_G})
\end{aligned}$$

Finally, we collect the terms

$$\begin{aligned}
\tilde{C}_1+C_2 &= \frac{1}{n}\sum_{i=1}^{n}\exp(\mu_i+\frac{1}{2}\sigma_i^2)\Phi(\frac{\mu_G-\ln L+\gamma_i}{\sigma_G}) \\
&\quad -K\Phi(\frac{\mu_G-\ln L}{\sigma_G})
\end{aligned}$$

from which (8.24) follows. ■

[Curran, 1994] does not mention any upper bounds, but [Rogers and Shi, 1995], whose method relies on conditioning on the Brownian Motion, use the inequalities

$$\begin{aligned}
0 &\leq E[U^+]-(E[U])^+ \\
&= \frac{1}{2}(E[|U|]-|E[U]|) \\
&\leq \frac{1}{2}E[|U-E[U]|] \\
&\leq \frac{1}{2}\sqrt{Var(U)} \qquad (8.25)
\end{aligned}$$

to form bounds similar to

$$\begin{aligned}
0 &\leq E[E[(A-K)^+|G]-(E[A-K|G])^+] \\
&\leq \frac{1}{2}E[\sqrt{Var(A-K|G)}] \qquad (8.26)
\end{aligned}$$

From (8.21) an upper bound on C_1 is then

$$\begin{aligned}
\bar{C}_1 &= \int_0^K (E[(A-K)|G=x])^+ g(x)dx \\
&\quad +\frac{1}{2}\int_0^K \sqrt{Var(A-K|G)}g(x)dx \qquad (8.27)
\end{aligned}$$

Due to time constraints we have not been able to pursue this upper bound any further.

8.1.3 Method overview

The Curran method can be summarized as follows

- Calculate the mean and the variance of $\ln G$ (the logarithm of the geometric average).
- For each averaging time t_i calculate the mean and the variance of $\ln S_{t_i}$ (the logarithm of the stock price at time t_i) and the covariance between $\ln G$ and $\ln S_{t_i}$.
- Use a numerical method to find the lower limit L of the integral in (8.21).
- Obtain a lower bound on the price of the Asian option by means of Theorem 17.

8.2 Numerical results

Table 8.1 contains the Curran lower bound (CLB) on the option price for different values of stock price volatility σ, strike price K, number of prices in the average n and maturity T. The Monte Carlo simulation has been carried out with 50,000 paths and the geometric option price as control variate, cf. Chapter 4. We have also included the approximation $\widetilde{L}$ of the lower limit L in the C_1 integral (8.21). Unless otherwise mentioned the stock dividend, the interest rate and the initial stock price have been fixed at

$$q = 0, \quad r = 0.05, \quad S_0 = 100 \tag{8.28}$$

The lower bound is extremely accurate, which we can see in the last column of Table 8.1. CLB and the Monte Carlo estimates only differ on the third or fourth decimal and CLB is inside the 95% confidence interval for all parameter values as seen in Figure 8.1 and Figure 8.2. These figures have also been used to illustrate that the Curran method generally is better than averaging the Levy price and the Milevsky and Posner price, cf. Chapter 6. Though, it is worth noting that MPLevy is inside the 95% confidence interval in most cases.

In Figure 8.2 we see that the lower bound is greater than the Monte Carlo estimate for most values of the strike price, which tells us two things. First of all the MC estimates are too low and secondly the Curran method is so accurate that even Monte Carlo simulation has problems matching it. We have run additional Monte Carlo simulations with 100,000 and 200,000 paths, which show that CLB is below the MC estimates except when the option is deep-in-the-money. CLB is still inside the 95% confidence interval and the relative differences are of magnitude less than 0.01 %, which should not cause any concern.

Table 8.1 shows that $\tilde{L}$ is close to K and decreasing in σ and T, which is in accordance with the differences between the geometric and arithmetic option prices seen in Figure 4.4 and Figure 4.5. The closer L (and hence $\tilde{L}$) is to K, the smaller $\tilde{C}_1$ is, which implies a more accurate lower bound $\tilde{C}$. We also see that the difference between $\tilde{L}$ and K is smallest for $K = 100 = S_0$ and it increases as the strike price moves away from the initial stock price.

8.2.1 Conclusion

We implemented the lower bound in [Curran, 1994] on the price of an Asian option and the result was very inspiring. The lower bound is so precise that it is the true price for all practical purposes. In contrast to other methods, the accuracy is retained even for large volatilities and long maturities as well as the whole range of strike prices.

The Curran method revealed that our Monte Carlo estimates from simulating with 50,000 paths were slightly too small, since the estimates were below the lower bound in some cases. Increasing the number of paths to 200,000 fixed this, but was also very time-consuming.

The lower bound was inside the 95% confidence interval for the Monte Carlo estimates for all parameter values. To put this into perspective we used the simpler one of the two approximations in [Curran, 1994] since it does not require numerical integration. Finally, we suggested how to obtain an upper bound on the option price inspired by [Rogers and Shi, 1995].

σ	**K**	**n**	**T**	**MC**	**sd**	**CLB**	$\tilde{\mathbf{L}}$	**CLB-MC**
0.05	100	12	1	2.93	0.0001	2.93	99.9856	0.0000
0.10	100	12	1	3.90	0.0004	3.90	99.9456	0.0000
0.15	100	12	1	5.01	0.0009	5.01	99.8791	0.0000
0.20	100	12	1	6.16	0.0016	6.16	99.7859	0.0003
0.25	100	12	1	7.31	0.0024	7.31	99.6662	0.0001
0.30	100	12	1	8.47	0.0035	8.47	99.5198	0.0001
0.35	100	12	1	9.64	0.0049	9.64	99.3467	0.0002
0.40	100	12	1	10.80	0.0065	10.80	99.1473	0.0003
0.45	100	12	1	11.96	0.0085	11.96	98.9212	0.0004
0.50	100	12	1	13.11	0.0107	13.12	98.6684	0.0008
0.20	100	12	**0.5**	4.11	0.0007	4.11	99.8931	0.0001
0.20	100	12	**1.0**	6.16	0.0016	6.16	99.7859	0.0003
0.20	100	12	**1.5**	7.84	0.0025	7.84	99.6786	0.0003
0.20	100	12	**2.0**	9.32	0.0034	9.32	99.5712	0.0002
0.20	100	12	**2.5**	10.66	0.0045	10.66	99.4637	0.0001
0.20	100	12	**3.0**	11.90	0.0055	11.90	99.3561	0.0002
0.20	100	12	**3.5**	13.05	0.0066	13.05	99.2483	0.0004
0.20	100	12	**4.0**	14.14	0.0078	14.14	99.1405	0.0007
0.20	100	12	**4.5**	15.16	0.0090	15.16	99.0325	0.0012
0.20	100	12	**5.0**	16.12	0.0102	16.13	98.9243	0.0019
0.20	100	**12**	1	6.16	0.0016	6.16	99.7859	0.0003
0.20	100	**24**	1	5.96	0.0016	5.96	99.7921	-0.0011
0.20	100	**36**	1	5.90	0.0016	5.89	99.7945	-0.0020
0.20	100	**48**	1	5.86	0.0016	5.86	99.7958	-0.0026
0.20	100	**60**	1	5.84	0.0016	5.84	99.7966	-0.0027
0.20	**70**	12	1	31.16	0.0015	31.16	69.1140	0.0009
0.20	**75**	12	1	26.42	0.0015	26.42	74.3369	0.0008
0.20	**80**	12	1	21.72	0.0015	21.72	79.5133	0.0003
0.20	**85**	12	1	17.16	0.0015	17.16	84.6450	0.0006
0.20	**90**	12	1	12.92	0.0015	12.92	89.7334	0.0007
0.20	**95**	12	1	9.19	0.0015	9.19	94.7799	0.0001
0.20	**100**	12	1	6.16	0.0016	6.16	99.7859	0.0003
0.20	**105**	12	1	3.87	0.0016	3.87	104.7530	0.0002
0.20	**110**	12	1	2.29	0.0015	2.29	109.6826	-0.0001
0.20	**115**	12	1	1.28	0.0015	1.28	114.5759	-0.0005
0.20	**120**	12	1	0.67	0.0013	0.67	119.4341	0.0004
0.20	**125**	12	1	0.34	0.0012	0.34	124.2585	0.0007
0.20	**130**	12	1	0.16	0.0010	0.16	129.0501	0.0012

Table 8.1: Curran lower bound (CLB) on the Asian option price for different values of volatility, strike price, number of prices in the average and maturity. MC: Monte Carlo estimate. sd: Standard deviation of MC estimator. $\tilde{L}$: Approximation of L in (8.21).

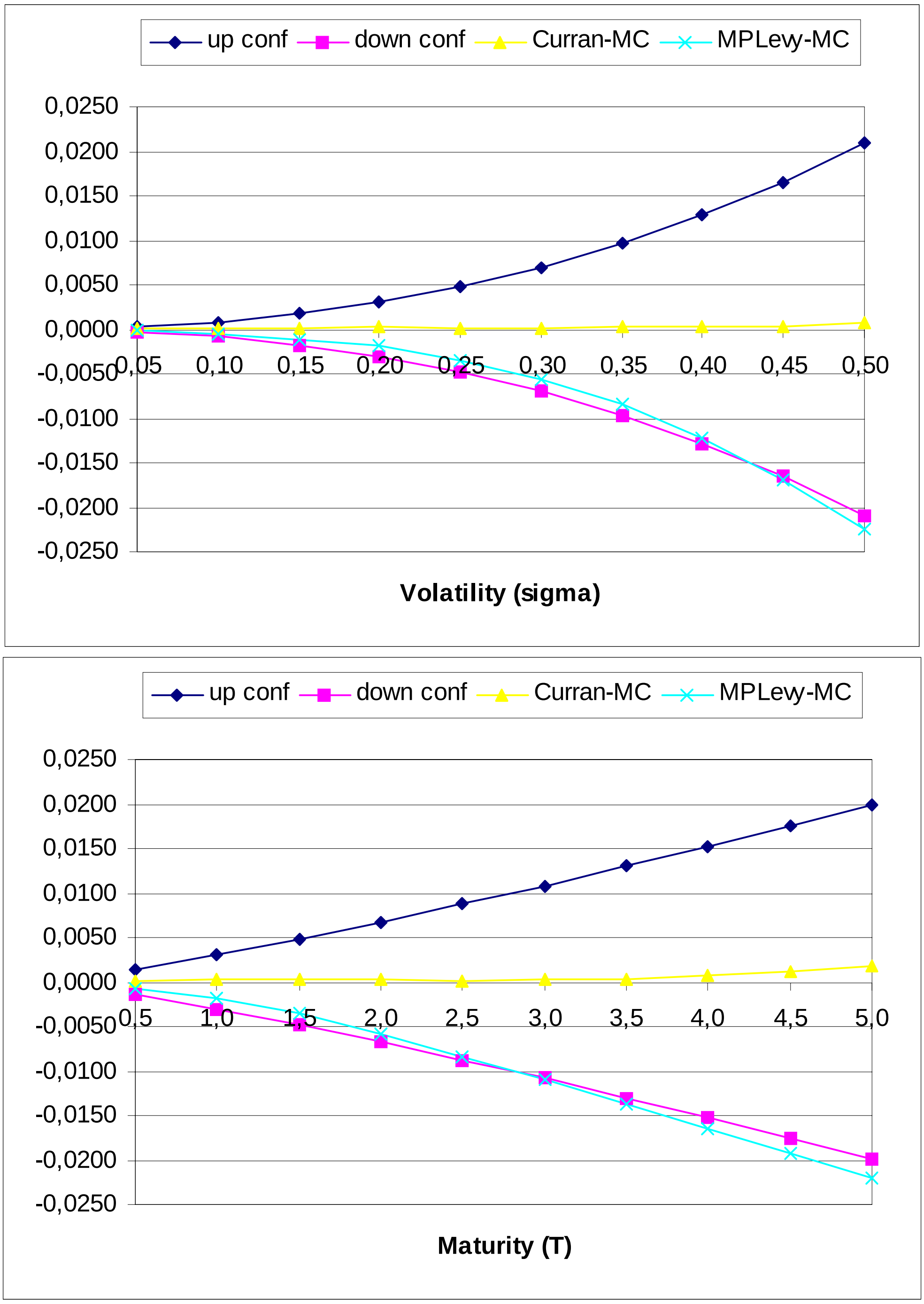


Figure 8.1: Curran lower bound on the Asian option price and 95% confidence interval for the Monte Carlo estimate. MPLevy: Average of Levy price and Milevsky and Posner price.

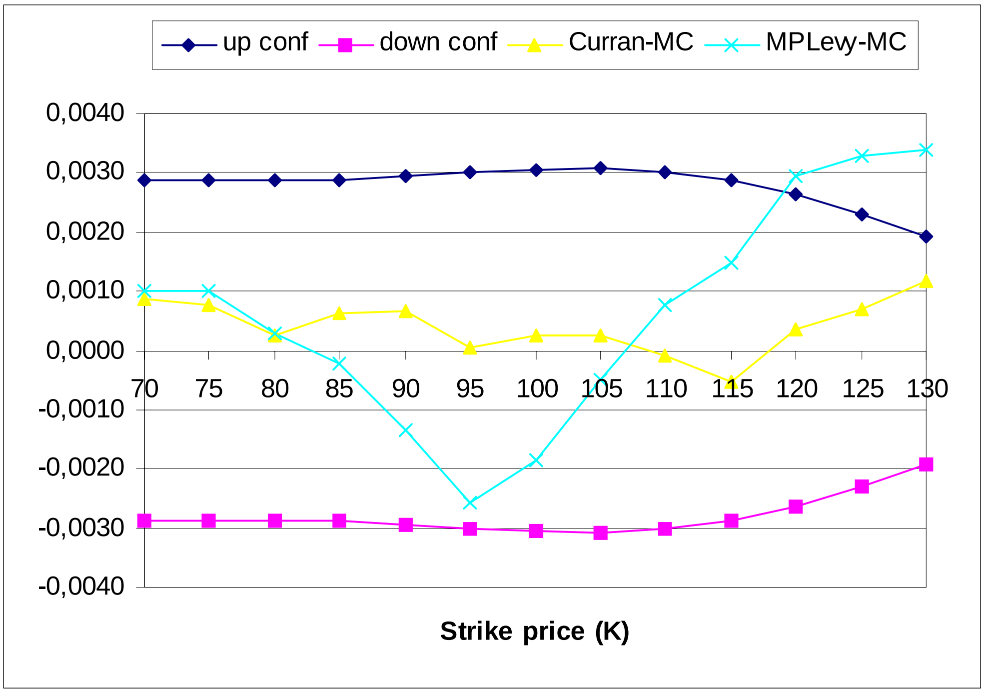


Figure 8.2: Curran lower bound on the Asian option price and 95% confidence interval for the Monte Carlo estimate. MPLevy: Average of Levy price and Milevsky and Posner price.

Chapter 9

Conclusion

Numerical results

In this thesis I have derived and implemented selected methods for pricing Asian options. First of all, I looked at Monte Carlo simulation with and without variance reduction methods. The numerical results led me to conclude that using the price of the geometric Asian option as control variate is by far the best way to price the arithmetic Asian option. This is due to the high correlation between the geometric average and the arithmetic average. The estimates of the standard method and the antithetic method had standard deviations so large that they could not be relied on, and reducing the standard deviations by increasing the number of paths was not computationally feasible. I also realized that the quality of the random numbers is very important and I expect that a more sophisticated random number generator can lead to better Monte Carlo estimates.

The pricing method in [Turnbull and Wakeman, 1991] performed very well for small stock price volatilities and short maturities. For large volatilities and maturities longer than a couple of years, the kurtosis correction in the generalized Edgeworth series expansion resulted in large price deviations from the Monte Carlo estimates. I studied the impact of removing correction terms in the pricing expression and I concluded that not adjusting for differences in skewness and kurtosis, which is equivalent to the method in [Levy, 1992], improved the accuracy in some cases.

Having used the lognormal distribution as approximation for the unknown distribution of the arithmetic average, I turned to the reciprocal gamma distribution suggested by [Milevsky and Posner, 1998]. The numerical results showed no improvement compared to the Levy method but the pricing errors of the two methods seemed to be negatively correlated. This gave me the idea to take the average of the prices, which turned out surprisingly well in terms of accuracy. I also applied the generalized Edgeworth series expansion in the Milevsky and Posner setup, which led to good results for short maturities. As in the Turnbull and Wakeman case, the pricing errors were substantial for maturities longer than a couple of years, and the accuracy was better with the lognormal distribution in general. Finally, I showed a considerable price difference between Asian options based on discrete and continuous averages.

The lognormal approximation in [Vorst, 1992] was only acceptable for small volatilities and in-the-money options with short maturities. I also derived a pricing formula for the geometric Asian option, which was used as a lower bound on the price of the arithmetic Asian option. The pricing bounds were very easy to calculate but they were too wide to constrain the price sufficiently.

The last pricing method was the lower bound in [Curran, 1994]. The numerical results were almost indistinguishable from the Monte Carlo estimates, so I concluded that the lower bound is the true price for all practical purposes. Increasing the stock price volatility and the maturity did not spoil the accuracy, since the lower bound was still inside the 95% confidence interval for the Monte Carlo estimates. The Curran method was actually more precise than Monte Carlo simulation with 50,000 paths, and it took more than 100,000 paths to get the Monte Carlo estimates above the lower bound for most values of the strike price.

Comparison of pricing methods

I have compared the methods as they appeared in the previous chapters and I shall summarize the conclusions here. It is difficult to rank the methods in terms of accuracy since none of them are uniformly better than any other method. Furthermore, many parameters can be varied, so I shall draw conclusions on a general purpose basis.

The Vorst method heavily underpriced out-of-the-money options and consequently it is my candidate for the least usable method. Then follow the Levy method and the Milevsky and Posner method due to their large variation and sensitivity to volatility and maturity. Next in line are the Turnbull and Wakeman method and its reciprocal gamma equivalent MP2, since they performed well in many cases, but unfortunately they were next to useless for long maturities. My idea of taking the average of the Levy price and the Milevsky and Posner price is the second best method. For a wide range of parameter values, it gave prices within the 95% confidence interval for the Monte Carlo estimates. The best method, in my opinion, is the Curran method, because the lower bound was very close to the Monte Carlo estimates even for large values of volatility and maturity.

Thus, the main conclusions of this thesis are:

- The lower bound in [Curran, 1994] on the price of an arithmetic Asian option is very accurate and reliable.
- The price of the geometric Asian option is a very suitable control variate for Monte Carlo simulation of the price of an arithmetic Asian option.

Computational aspects

An in-depth discussion of the programming is not appropriate in my point of view, but I would like to add a few remarks. For anyone with some programming experience, all the methods should be easy to implement by means of this thesis. Extending the code to a more generalized setting should not pose a problem and neither should improvements to reduce memory usage and computational time. The priority for me has been to write code that is usable and reliable.

The Monte Carlo simulation is very time-consuming, especially when using many paths to obtain a high level of precision. I used a very simple numerical method in the implementation of the Curran method, so obtaining one bound takes a couple of seconds. The rest of the methods only use a fraction of a second per price, which implies that all methods except Monte Carlo simulation can be used in real time to support trading.

Future research

The basic Black-Scholes model used in this thesis enables many generalizations. It could be interesting to investigate if the conclusions made also apply to setups with e.g. pricing in the averaging period, flexible and floating Asian options, non-lognormal underlying assets and a stochastic term structure of interest rates.

Calculation of accurate hedging parameters in relation to the pricing methods is another important direction for future research, since Asian options are often used for hedging purposes.

Finally, it would be advantageous to find out if the upper bound mentioned in Chapter 8 is as good as the lower bound.

Appendix A

Monte Carlo simulation

A.1 Visual Basic code

The code can also be downloaded from the website http://www.lbn.dk/master/.

A.1.1 Standard Monte Carlo

```
Function GenerateRandomNumbers() As Variant

Dim pi As Double, a As Double, c As Double
Dim n As Long, n_max As Long
Dim I() As Double
Dim X() As Double

pi = 4 * Atn(1)
a = 7 ^5
c = 2 ^31 - 1
n_max = 6000000           '# random numbers

ReDim I(n_max)
ReDim X(n_max)

I(0) = 1773             'seed for the linear congruential generator
For n = 1 To n_max      'I() divided by c is iid U(0,1)
  I(n) = c * (a * I(n - 1) / c - Int(a * I(n - 1) / c))     _
  'a times I() modulo c
Next n

For n = 1 To n_max - 1 Step 2
```

```
  'Box-Muller method: X() is iid N(0,1)
  X(n) = Sqr(-2 * Log(I(n) / c)) * Cos(2 * pi * (I(n + 1) / c))
  X(n + 1) = Sqr(-2 * Log(I(n) / c)) * Sin(2 * pi * (I(n + 1) / c))
Next n

Open ''c:\dokumenter\montecarlo\random_normal_seed_1773.txt'' _
  For Output As #1
For n = 1 To n_max
  Write #1, X(n)
Next n
Close #1

End Function

Function MonteCarlo(sigma As Double, K As Double, m_ave As Integer, _
  q As Double, r As Double, S_0 As Double, T As Double) As Variant

Dim n As Long, n_path As Long, m As Long
Dim h As Double
Dim Z() As Double, S() As Double, AV() As Double
Dim asian() As Double, european() As Double
Dim V As Double, W As Double
Dim sdA As Double, sdE As Double
Dim output() As Double

'sigma: volatility
'K:     strike price
'm_ave: # averaging times
'q:     stock dividend
'r:     interest rate
'S_0:   stock price at time 0
'T:     time to maturity

n_path = 50000  '# paths (replications) in the Monte Carlo simulation
h = T / m_ave   'time between successive averaging times

ReDim Z(n_path, m_ave)
ReDim S(n_path, m_ave)
ReDim AV(n_path)
ReDim asian(n_path)
ReDim european(n_path)
```

```
ReDim output(3)

Open ''c:\dokumenter\montecarlo\random_normal_seed_1773.txt'' _
  For Input As #2
For m = 1 To m_ave
  For n = 1 To n_path
    Input #2, Z(n, m)
  Next n
Next m
Close #2

For n = 1 To n_path       'stock price simulation
 S(n, 0) = S_0
 AV(n) = 0
 For m = 1 To m_ave
   S(n, m) = S(n, m - 1) * Exp((r - q - sigma ^2 / 2) * h + _
     sigma * Sqr(h) * Z(n, m)) 'simulated path
   AV(n) = AV(n) + S(n, m)
 Next m
 AV(n) = AV(n) / m_ave        'average stock price
Next n

For n = 1 To n_path             'option price simulation
  asian(n) = Exp(-r * T) * MaxFunc(AV(n) - K, 0)
  european(n) = Exp(-r * T) * MaxFunc(S(n, m_ave) - K, 0)
Next n

V = 0
W = 0
For n = 1 To n_path             'Monte Carlo simulation
  V = asian(n) + V
  W = european(n) + W
Next n
V = V / n_path          'Asian option price
W = W / n_path          'European option price

sdA = 0
sdE = 0
For n = 1 To n_path
  sdA = (asian(n) - V) ^2 + sdA
  sdE = (european(n) - W) ^2 + sdE
```

```
Next n
sdA = Sqr(sdA / (n_path - 1))  'standard deviation _
  (square root of sample variance)
sdE = Sqr(sdE / (n_path - 1))

output(0) = V
output(1) = sdA / Sqr(n_path)  'standard deviation of estimator
output(2) = W
output(3) = sdE / Sqr(n_path)

MonteCarlo = output()

End Function

Function MaxFunc(number1 As Double, number2 As Double) As Double

If number1 > number2 Then
  MaxFunc = number1
ElseIf number1 <= number2 Then
  MaxFunc = number2
Else
  Stop
End If

End Function

Function BlackScholes(sigma, K, q, r, Su, T, u) As Double
Dim d1 As Double, d2 As Double

d1 = (Log(Su / K) + (r - q + sigma ^2 / 2) * (T - u)) / _
  (sigma * Sqr(T - u))
d2 = d1 - sigma * Sqr(T - u)

BlackScholes = Su * WorksheetFunction.NormSDist(d1) - _
  Exp(-r * (T - u)) * K * WorksheetFunction.NormSDist(d2)

End Function
```

A.1.2 Antithetic method

```
Function MonteCarloAntithetic(sigma As Double, K As Double, _
```

```
  m_ave As Integer, q As Double, r As Double, S_0 As Double, _
  T As Double) As Variant

Dim n As Long, n_path As Long, m As Long
Dim h As Double
Dim Z() As Double, S() As Double, AV() As Double
Dim asian() As Double, european() As Double
Dim V As Double, W As Double
Dim sdA As Double, sdE As Double
Dim output() As Double

'sigma: volatility
'K:     strike price
'm_ave: # averaging times
'q:     stock dividend
'r:     interest rate
'S_0:   stock price at time 0
'T:     time to maturity

n_path = 50000  '# paths (replications) in the Monte Carlo simulation
h = T / m_ave   'time between successive averaging times

ReDim Z(n_path, m_ave)
ReDim S(2 * n_path, m_ave)
ReDim AV(2 * n_path)
ReDim asian(2 * n_path)
ReDim european(2 * n_path)
ReDim output(3)

Open ''c:\dokumenter\montecarlo\random_normal_seed_1773.txt'' _
  For Input As #2
For m = 1 To m_ave
  For n = 1 To n_path
    Input #2, Z(n, m)
  Next n
Next m
Close #2

'stock price simulation
For n = 1 To n_path
 S(n, 0) = S_0
```

```
 S(n + n_path, 0) = S_0
 AV(n) = 0
 AV(n + n_path) = 0
 For m = 1 To m_ave
   S(n, m) = S(n, m - 1) * Exp((r - q - sigma ^2 / 2) * h + _
     sigma * Sqr(h) * Z(n, m)) 'simulated path
   S(n + n_path, m) = S(n + n_path, m - 1) * Exp((r - q - sigma ^2 _
     / 2) * h + sigma * Sqr(h) * (-Z(n, m))) 'antithetic path
   AV(n) = AV(n) + S(n, m)
   AV(n + n_path) = AV(n + n_path) + S(n + n_path, m)
 Next m
 AV(n) = AV(n) / m_ave       'average stock price
 AV(n + n_path) = AV(n + n_path) / m_ave
Next n

'option price simulation
For n = 1 To 2 * n_path
  asian(n) = Exp(-r * T) * MaxFunc(AV(n) - K, 0)
  european(n) = Exp(-r * T) * MaxFunc(S(n, m_ave) - K, 0)
Next n

V = 0
W = 0
For n = 1 To 2 * n_path           'Monte Carlo simulation
  V = asian(n) + V
  W = european(n) + W
Next n
V = V / (2 * n_path)        'Asian option price
W = W / (2 * n_path)        'European option price

sdA = 0
sdE = 0
For n = 1 To n_path
  sdA = ((asian(n) + asian(n + n_path)) / 2 - V) ^2 + sdA
  sdE = ((european(n) + european(n + n_path)) / 2 - W) ^2 + sdE
Next n
sdA = Sqr(sdA / (n_path - 1)) 'standard deviation _
  (square root of sample variance)
sdE = Sqr(sdE / (n_path - 1))

output(0) = V
```

```
output(1) = sdA / Sqr(n_path)  'standard deviation of estimator
output(2) = W
output(3) = sdE / Sqr(n_path)

MonteCarloAntithetic = output()

End Function
```

A.1.3 Control variates

```
Function MonteCarloControl(sigma As Double, K As Double, _
  m_ave As Integer, q As Double, r As Double, S_0 As Double, _
  T As Double) As Variant

Dim n As Long, n_path As Long, m As Long
Dim h As Double
Dim Z() As Double, S() As Double, AV() As Double, AVgeo() As Double
Dim asian() As Double, european() As Double, asian_geo() As Double, _
  asian_new() As Double
Dim V As Double, W As Double, Vgeo As Double, Vnew As Double, _
  Vgeo_true As Double
Dim sdA As Double, sdE As Double, sdAgeo As Double, sdAnew As Double
Dim output() As Double

'sigma: volatility
'K:     strike price
'm_ave: # averaging times
'q:     stock dividend
'r:     interest rate
'S_0:   stock price at time 0
'T:     time to maturity

n_path = 50000  '# paths (replications) in the Monte Carlo simulation
h = T / m_ave   'time between successive averaging times

ReDim Z(n_path, m_ave)
ReDim S(n_path, m_ave)
ReDim AV(n_path)
ReDim AVgeo(n_path)
ReDim asian(n_path)
ReDim european(n_path)
```

```
ReDim asian_geo(n_path)
ReDim asian_new(n_path)
ReDim output(8)

Open ''c:\dokumenter\montecarlo\random_normal_seed_1773.txt'' _
  For Input As #2
For m = 1 To m_ave
  For n = 1 To n_path
    Input #2, Z(n, m)
  Next n
Next m
Close #2

'stock price simulation
For n = 1 To n_path
 S(n, 0) = S_0
 AV(n) = 0
 AVgeo(n) = 1
 For m = 1 To m_ave
   S(n, m) = S(n, m - 1) * Exp((r - q - sigma ^2 / 2) * h + _
     sigma * Sqr(h) * Z(n, m)) 'simulated path
   AV(n) = AV(n) + S(n, m)
   AVgeo(n) = AVgeo(n) * S(n, m)
 Next m
 AV(n) = AV(n) / m_ave        'average stock price
 AVgeo(n) = AVgeo(n) ^(1 / m_ave)  'geometric average
Next n

Vgeo_true = BS_geometric(sigma, K, m_ave, q, r, S_0, T) _
  'Analytical solution

'option price simulation
For n = 1 To n_path
  asian(n) = Exp(-r * T) * MaxFunc(AV(n) - K, 0)
  asian_geo(n) = Exp(-r * T) * MaxFunc(AVgeo(n) - K, 0)
  'european(n) = Exp(-r * T) * MaxFunc(S(n, m_ave) - K, 0)
  asian_new(n) = asian(n) - asian_geo(n) + Vgeo_true
Next n

'Monte Carlo simulation
V = 0
```

```
Vgeo = 0
'W = 0
For n = 1 To n_path
  V = asian(n) + V
  Vgeo = asian_geo(n) + Vgeo
  'W = european(n) + W
Next n
V = V / n_path          'Asian option price
Vgeo = Vgeo / n_path    'Geometric Asian option price
'W = W / n_path          'European option price

Vnew = V - Vgeo + Vgeo_true         _
  'The control variate is the geometric option

'sdA = 0
sdAgeo = 0
'sdE = 0
sdAnew = 0
For n = 1 To n_path
  'sdA = (asian(n) - V) ^2 + sdA
  sdAgeo = (asian_geo(n) - Vgeo) ^2 + sdAgeo
  'sdE = (european(n) - W) ^2 + sdE
  sdAnew = (asian_new(n) - Vnew) ^2 + sdAnew
Next n
'sdA = Sqr(sdA / (n_path - 1))  'standard deviation _
  (square root of sample variance)
sdAgeo = Sqr(sdAgeo / (n_path - 1))
'sdE = Sqr(sdE / (n_path - 1))
sdAnew = Sqr(sdAnew / (n_path - 1))

output(0) = Vnew
output(1) = sdAnew / Sqr(n_path)    'standard deviation of estimator
output(2) = Vgeo
output(3) = sdAgeo / Sqr(n_path)
output(4) = Vgeo_true
'output(5) = V
'output(6) = sdA / Sqr(n_path)
'output(7) = W
'output(8) = sdE / Sqr(n_path)

MonteCarloControl = output()
```

```
End Function

Function BS_geometric(sigma, K, m_ave, q, r, S_0, T) As Double
Dim h As Double, mg As Double, sg As Double, d1 As Double, _
  d2 As Double

h = T / m_ave

mg = Log(S_0) + (r - q - sigma ^2 / 2) * (T + h) / 2
sg = Sqr(sigma ^2 * h * (2 * m_ave + 1) * (m_ave + 1) / (6 * m_ave))

d1 = (mg - Log(K) + sg ^2) / sg
d2 = d1 - sg

BS_geometric = Exp(-r * T) * (Exp(mg + sg ^2 / 2) * _
  WorksheetFunction.NormSDist(d1) - _
  K * WorksheetFunction.NormSDist(d2))

End Function
```

Appendix B

Turnbull and Wakeman

B.1 Moments

To calculate the moments of the true distribution of the average, we use

$$E[L_i^m] = E[(1 + R_i L_{i+1})^m], \quad i = 1, 2, ..., n, \quad m = 1, 2, ... \tag{B.1}$$

For real numbers a and b

$$(a+b)^2 = a^2 + b^2 + 2ab \tag{B.2}$$
$$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2 \tag{B.3}$$
$$(a+b)^4 = a^4 + b^4 + 4a^3b + 4ab^3 + 6a^2b^2 \tag{B.4}$$

The first four moments of L_i are then

$$E[L_i] = E[1 + R_i L_{i+1}] \tag{B.5}$$

$$\begin{aligned} E[L_i^2] &= E[(1 + R_i L_{i+1})^2] \\ &= E[1 + R_i^2 L_{i+1}^2 + 2R_i L_{i+1}] \\ &= 1 + E[R_i^2]E[L_{i+1}^2] + 2E[R_i]E[L_{i+1}] \end{aligned} \tag{B.6}$$

$$\begin{aligned} E[L_i^3] &= E[(1 + R_i L_{i+1})^3] \\ &= E[1 + R_i^3 L_{i+1}^3 + 3R_i L_{i+1} + 3R_i^2 L_{i+1}^2] \\ &= 1 + E[R_i^3]E[L_{i+1}^3] + 3E[R_i^2]E[L_{i+1}^2] + 3E[R_i]E[L_{i+1}] \end{aligned} \tag{B.7}$$

$$\begin{aligned} E[L_i^4] &= E[(1 + R_i L_{i+1})^4] \\ &= E[1 + R_i^4 L_{i+1}^4 + 4R_i L_{i+1} + 4R_i^3 L_{i+1}^3 + 6R_i^2 L_{i+1}^2] \\ &= 1 + E[R_i^4]E[L_{i+1}^4] + 4E[R_i^3]E[L_{i+1}^3] + 6E[R_i^2]E[L_{i+1}^2] \\ &\quad +4E[R_i]E[L_{i+1}] \end{aligned} \tag{B.8}$$

B.2 Cumulants

For real numbers a and b

$$\begin{aligned}(a-b)^2 &= a^2+b^2-2ab\\ (a-b)^3 &= a^3-b^3-3a^2b+3ab^2\\ (a-b)^4 &= a^4+b^4-4a^3b-4ab^3+6a^2b^2\end{aligned}$$

Let Y have distribution F. Then the cumulants are

$$\kappa_1(F) = E[Y] \tag{B.9}$$

$$\begin{aligned}\kappa_2(F) &= E[(Y-E[Y])^2]\\ &= E[Y^2+E[Y]^2-2YE[Y]]\\ &= E[Y^2]-E[Y]^2\end{aligned} \tag{B.10}$$

$$\begin{aligned}\kappa_3(F) &= E[(Y-E[Y])^3]\\ &= E[Y^3-E[Y]^3-3Y^2E[Y]+3YE[Y]^2]\\ &= E[Y^3]-E[Y]^3-3E[Y^2]E[Y]+3E[Y]^3\\ &= E[Y^3]+2E[Y]^3-3E[Y^2]E[Y]\end{aligned} \tag{B.11}$$

$$\begin{aligned}\kappa_4(F) &= E[(Y-E[Y])^4]-3(\kappa_2(F))^2\\ &= E[Y^4+E[Y]^4-4Y^3E[Y]-4YE[Y]^3+6Y^2E[Y]^2]\\ &\quad -3(E[Y^2]-E[Y]^2)^2\\ &= E[Y^4]+E[Y]^4-4E[Y^3]E[Y]-4E[Y]^4+6E[Y^2]E[Y]^2\\ &\quad -3E[Y^2]^2-3E[Y]^4+6E[Y^2]E[Y]^2\\ &= E[Y^4]-6E[Y]^4-4E[Y^3]E[Y]+12E[Y^2]E[Y]^2\\ &\quad -3E[Y^2]^2\end{aligned} \tag{B.12}$$

[Turnbull and Wakeman, 1991] and [Jarrow and Rudd, 1982] are inconsistent regarding the definition of κ_4. The first two authors subtract $3\kappa_2(F)$ instead of $3(\kappa_2(F))^2$, which is probably due to a typographical error. Numerically, this doesn't make any difference in our setup, because we have set the first two moments equal in the two distributions.

B.3 Moment matching

Matching the moments of the approximating lognormal distribution to the true distribution requires us to solve the following system of equations

$$\begin{aligned}
&\alpha_1(F) = \alpha_1(G)\\
&\alpha_2(F) = \alpha_2(G)\\
&\Updownarrow\\
&\alpha_1(F) = \exp(\nu + \tfrac{1}{2}\lambda^2)\\
&\alpha_2(F) = \exp(2\nu + 2\lambda^2)\\
&\Updownarrow\\
&\ln\alpha_1(F) = \nu + \tfrac{1}{2}\lambda^2\\
&\ln\alpha_2(F) = 2\nu + 2\lambda^2\\
&\Updownarrow\\
&\ln\alpha_1(F) = \nu + \tfrac{1}{2}\lambda^2\\
&\ln\alpha_2(F) = 2(\ln\alpha_1(F) - \tfrac{1}{2}\lambda^2) + 2\lambda^2\\
&\Updownarrow\\
&\ln\alpha_1(F) = \nu + \tfrac{1}{2}\lambda^2\\
&\lambda^2 = \ln\alpha_2(F) - 2\ln\alpha_1(F)\\
&\Updownarrow\\
&\ln\alpha_1(F) = \nu + \tfrac{1}{2}(\ln\alpha_2(F) - 2\ln\alpha_1(F))\\
&\lambda^2 = \ln\alpha_2(F) - 2\ln\alpha_1(F)\\
&\Updownarrow\\
&\nu = 2\ln\alpha_1(F) - \tfrac{1}{2}\ln\alpha_2(F)\\
&\lambda^2 = \ln\alpha_2(F) - 2\ln\alpha_1(F)
\end{aligned}$$

B.4 Lognormal density and derivatives

The density function $g(x)$ for the lognormal distribution with parameters ν and λ^2 is given by

$$g(x) = \frac{1}{\lambda\sqrt{2\pi}}\frac{1}{x}\exp(\frac{-(\nu - \ln x)^2}{2\lambda^2}), \quad x > 0 \tag{B.13}$$

The first derivative of g with respect to x is

$$\begin{aligned}
\frac{dg}{dx}(x) &= \frac{1}{\lambda\sqrt{2\pi}}\frac{-1}{x^2}\exp(\frac{-(\nu - \ln x)^2}{2\lambda^2})\\
&\quad + \frac{1}{\lambda\sqrt{2\pi}}\frac{1}{x}\exp(\frac{-(\nu - \ln x)^2}{2\lambda^2})\frac{-1}{2\lambda^2}2(\nu - \ln x)\frac{-1}{x}\\
&= g(x)(\frac{-1}{x} + \frac{1}{x}\frac{1}{\lambda^2}(v - \ln x)\\
&= g(x)\frac{1}{x}(\frac{\nu - \ln x}{\lambda^2} - 1), \quad x > 0 \qquad \text{(B.14)}
\end{aligned}$$

Set

$$\phi(x) = \frac{\nu - \ln x}{\lambda^2} - 1, \quad x > 0 \tag{B.15}$$

The second derivative is then

$$\begin{aligned}
\frac{d^2 g}{dx^2}(x) &= (g(x)\frac{1}{x}\phi(x))' \\
&= g'(x)\frac{1}{x}\phi(x) + g(x)(\frac{-1}{x^2}\phi(x) + \frac{1}{x}\phi'(x)) \\
& \quad g(x)\frac{1}{x^2}(\phi(x))^2 - g(x)\frac{1}{x^2}\phi(x) + g(x)\frac{1}{x}\frac{1}{\lambda^2}\frac{-1}{x} \\
&= g(x)\frac{1}{x^2}(\phi(x)(\phi(x) - 1) - \frac{1}{\lambda^2}) \\
&= g(x)\frac{1}{x^2}((\frac{\nu - \ln x}{\lambda^2} - 1)(\frac{\nu - \ln x}{\lambda^2} - 2) - \frac{1}{\lambda^2}), \ x > 0 \quad \text{(B.16)}
\end{aligned}$$

B.5 Visual Basic code

```
Function TurnbullWakeman(sigma As Double, K As Double, _
  n As Integer, q As Double, r As Double, S_0 As Double, _
  T As Double) As Variant
Dim sigSq As Double, h As Double, k2 As Double
Dim R1 As Double, R2 As Double, R3 As Double, R4 As Double
Dim L1 As Double, L2 As Double, L3 As Double, L4 As Double
Dim j As Integer
Dim Y1 As Double, Y2 As Double, Y3 As Double, Y4 As Double
Dim A1 As Double, A2 As Double, A3 As Double, A4 As Double
Dim my_approx As Double, sigSq_approx As Double
Dim c2 As Double, c3 As Double, c4 As Double
Dim d1 As Double, d2 As Double
Dim price As Double
Dim term1 As Double, term2 As Double, term3 As Double
Dim Price2 As Double, Price3 As Double, PriceAllTerms As Double
Dim output(8) As Double

'sigma: volatility
'K:     strike price
'n:     # averaging times
'q:     stock dividend
'r:     interest rate
```

```
'S_0:   stock price at time 0
'T:     time to maturity

sigSq = sigma ^2
h = T / n           'time increment between averaging dates
k2 = K * n / S_0  'scaled strike price - (lower case) k in TW article

'PART 1 - calculate moments of true distribution
'i=T-(n-1) in TW article
R1 = LogMoment(1, (r - q - sigSq / 2) * h, sigSq * h)
R2 = LogMoment(2, (r - q - sigSq / 2) * h, sigSq * h)
R3 = LogMoment(3, (r - q - sigSq / 2) * h, sigSq * h)
R4 = LogMoment(4, (r - q - sigSq / 2) * h, sigSq * h)
L1 = 1      'L(T+1)=L(i+n)=1
L2 = 1
L3 = 1
L4 = 1

For j = (n - 1) To 1 Step -1    'time i+j, i.e. time T to time T-n+2
  'Priority L4,L3,L2,L1 important!
  L4 = 1 + R4 * L4 + 4 * R3 * L3 + 6 * R2 * L2 + 4 * R1 * L1
  L3 = 1 + R3 * L3 + 3 * R2 * L2 + 3 * R1 * L1
  L2 = 1 + R2 * L2 + 2 * R1 * L1
  L1 = 1 + R1 * L1
Next j

'Moments of true distribution
Y1 = R1 * L1
Y2 = R2 * L2
Y3 = R3 * L3
Y4 = R4 * L4

'PART 2 - calculate moments of approximating distribution
'match moments of approx. dist. to true dist.
my_approx = 2 * Log(Y1) - Log(Y2) / 2
sigSq_approx = Log(Y2) - 2 * Log(Y1)

'Moments of approximating distribution
A1 = Y1         'per construction
A2 = Y2         'per construction
A3 = LogMoment(3, my_approx, sigSq_approx)
```

```
A4 = LogMoment(4, my_approx, sigSq_approx)

'PART 3 - calculate option price
c2 = Y2 - Y1 ^2 - (A2 - A1 ^2)
c3 = Y3 + 2 * Y1 ^3 - 3 * Y2 * Y1
c3 = c3 - (A3 + 2 * A1 ^3 - 3 * A2 * A1)
c4 = Y4 - 6 * Y1 ^4 - 4 * Y3 * Y1 + 12 * Y2 * Y1 ^2 - 3 * Y2 ^2
c4 = c4 - (A4 - 6 * A1 ^4 - 4 * A3 * A1 + 12 * A2 * A1 ^2 - 3 * A2 ^2)
c4 = c4 + 3 * c2 ^2

d1 = (-Log(k2) + my_approx + sigSq_approx) / Sqr(sigSq_approx)
d2 = d1 - Sqr(sigSq_approx)

price = Exp(my_approx + sigSq_approx / 2) * _
  WorksheetFunction.NormSDist(d1) - _
  k2 * WorksheetFunction.NormSDist(d2)
'price = price + c2 / 2 * LogDensity(0, k2, my_approx, sigSq_approx)
'price = price - c3 / 6 * LogDensity(1, k2, my_approx, sigSq_approx)
'price = price + c4 / 24 * LogDensity(2, k2, my_approx, sigSq_approx)
price = price * Exp(-r * T) * S_0 / n

'term1=0 in this setup since c2=0
term1 = c2 / 2 * LogDensity(0, k2, my_approx, sigSq_approx) * _
  Exp(-r * T) * S_0 / n
term2 = -c3 / 6 * LogDensity(1, k2, my_approx, sigSq_approx) * _
  Exp(-r * T) * S_0 / n
term3 = c4 / 24 * LogDensity(2, k2, my_approx, sigSq_approx) * _
  Exp(-r * T) * S_0 / n

Price2 = price + term1
Price3 = price + term1 + term2
PriceAllTerms = price + term1 + term2 + term3

output(0) = PriceAllTerms   'TW price
output(1) = Price3   'No correction for diff. in 4. cumulants
output(2) = Price2   'Levy price, no corr. for diff. in cumulants
output(3) = term2    'correction for diff. in 3. cumulants
output(4) = term3    'correction for diff. in 4. cumulants
output(5) = c3 / 6
output(6) = LogDensity(1, k2, my_approx, sigSq_approx)
output(7) = c4 / 24
```

```
output(8) = LogDensity(2, k2, my_approx, sigSq_approx)

TurnbullWakeman = output()

End Function

Function LogMoment(m As Integer, my As Double, sigSq As Double) _
  As Double
'Returns the m'th moment of logN(my, sigSq)

LogMoment = Exp(my * m + sigSq * m ^2 / 2)

End Function

Function LogDensity(deriv As Integer, x As Double, my As Double, _
  sigSq As Double) As Double
'Density function (and derivatives) of logN(my, sigSq)

Dim pi As Double, frac As Double, f As Double

pi = 4 * Atn(1)
frac = (my - Log(x)) / sigSq
f = 1 / (Sqr(2 * pi * sigSq) * x) * Exp(-(my - Log(x)) ^2 / (2 * sigSq))

If deriv = 0 Then
'lognormal density f(x)
LogDensity = f
ElseIf deriv = 1 Then
'1. order derivative f'(x)
LogDensity = f / x * (frac - 1)
ElseIf deriv = 2 Then
'2. order derivative f''(x)
LogDensity = f / x ^2 * ((frac - 1) * (frac - 2) - 1 / sigSq)
Else
  Stop
End If

End Function
```

Appendix C

Milevsky and Posner

C.1 Gamma distribution

Proposition 9 states that

$$G(x|\alpha,\beta) = G(\frac{x}{\beta}|\alpha,1), \quad \forall x>0 \tag{C.1}$$

$$g(x|\alpha,\beta) = \frac{1}{\beta}g(\frac{x}{\beta}|\alpha,1), \quad \forall x>0 \tag{C.2}$$

$$g(x|\alpha,\beta) = \frac{x}{\beta(\alpha-1)}g(x|\alpha-1,\beta), \quad \forall x>0, \quad \forall \alpha>1 \tag{C.3}$$

Proof of Proposition 9. Using the definition of the density function

$$\begin{aligned}
G(x|\alpha,\beta) &= \int_0^x g(u|\alpha,\beta)du \\
&= \int_0^x \Gamma(\alpha)^{-1}\beta^{-\alpha}u^{\alpha-1}\exp(-\beta^{-1}u)du \\
&= \int_0^{\frac{x}{\beta}} \Gamma(\alpha)^{-1}\beta^{-\alpha}\beta^{\alpha-1}v^{\alpha-1}\exp(-v)\beta dv \\
&= \int_0^{\frac{x}{\beta}} \Gamma(\alpha)^{-1}v^{\alpha-1}\exp(-v)dv \\
&= G(\frac{x}{\beta}|\alpha,1), \quad \forall x>0
\end{aligned}$$

where we have made the change of variables $v=\frac{u}{\beta}$ so $\frac{dv}{du}=\frac{1}{\beta}$. This proves (C.1). (C.2) follows from the above since

$$\begin{aligned}
g(x|\alpha,\beta) &= \frac{d}{dx}(G(x|\alpha,\beta)) = \frac{d}{dx}(G(\frac{x}{\beta}|\alpha,1)) \\
&= \frac{1}{\beta}g(\frac{x}{\beta}|\alpha,1), \quad \forall x>0
\end{aligned}$$

For $\alpha > 1$

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

which we use to prove (C.3)

$$\begin{aligned}
g(x|\alpha,\beta) &= \Gamma(\alpha)^{-1}\beta^{-\alpha}x^{\alpha-1}\exp(-\beta^{-1}x) \\
&= \frac{1}{(\alpha-1)\Gamma(\alpha-1)}\frac{1}{\beta\beta^{\alpha-1}}xx^{\alpha-2}\exp(-\beta^{-1}x) \\
&= \frac{x}{\beta(\alpha-1)}\Gamma(\alpha-1)^{-1}\beta^{-(\alpha-1)}x^{\alpha-2}\exp(-\beta^{-1}x) \\
&= \frac{x}{\beta(\alpha-1)}g(x|\alpha-1,\beta), \quad \forall x > 0, \quad \forall \alpha > 1
\end{aligned}$$

■

C.2 Reciprocal gamma distribution

C.2.1 Moments

Let $X \sim \Gamma(\alpha, \beta)$ and $Y = \frac{1}{X}$. Y is reciprocally gamma distributed by definition and its moments of order $m < \alpha$ are

$$\begin{aligned}
E[Y^m] &= E[\frac{1}{X^m}] = \int_0^\infty \frac{1}{x^m}g(x|\alpha,\beta)dx \\
&= \int_0^\infty \frac{1}{x^{m-1}}\frac{1}{\beta(\alpha-1)}g(x|\alpha-1,\beta)dx \\
&= \int_0^\infty \frac{1}{x^{m-2}}\frac{1}{\beta^2(\alpha-1)(\alpha-2)}g(x|\alpha-2,\beta)dx \\
&= \ldots \\
&= \int_0^\infty \frac{1}{\beta^m(\alpha-1)(\alpha-2)\ldots(\alpha-m)}g(x|\alpha-m,\beta)dx \\
&= \frac{1}{\beta^m(\alpha-1)(\alpha-2)\ldots(\alpha-m)}\int_0^\infty g(x|\alpha-m,\beta)dx \\
&= \frac{1}{\beta^m(\alpha-1)(\alpha-2)\ldots(\alpha-m)} \qquad \text{(C.4)}
\end{aligned}$$

where we have used (C.3) successively.

The first two (positive) moments of the reciprocal gamma distribution are, cf. (C.4)

$$M_1 = \frac{1}{\beta(\alpha-1)} \qquad \text{(C.5)}$$

and

$$M_2 = \frac{1}{\beta^2(\alpha-1)(\alpha-2)} \tag{C.6}$$

We want to solve for α and β in these equations. (C.5) and (C.6) can be written as

$$\beta = \frac{1}{M_1(\alpha-1)} \tag{C.7}$$

and

$$\beta^2 = \frac{1}{M_2(\alpha-1)(\alpha-2)} \tag{C.8}$$

Inserting (C.7) into (C.8) gives

$$\frac{1}{M_1^2(\alpha-1)^2} = \frac{1}{M_2(\alpha-1)(\alpha-2)}$$

$$\Updownarrow$$

$$M_1^2(\alpha-1) = M_2(\alpha-1) - M_2$$

$$\Updownarrow$$

$$(M_1^2 - M_2)(\alpha-1) = -M_2$$

$$\Updownarrow$$

$$(\alpha-1) = \frac{M_2}{M_2-M_1^2}$$

$$\Updownarrow$$

$$\alpha = \frac{2M_2-M_1^2}{M_2-M_1^2}$$

Inserting the last line but one into (C.7) gives

$$\beta = (M_1\frac{M_2}{M_2-M_1^2})^{-1} = \frac{M_2-M_1^2}{M_1M_2} \tag{C.9}$$

All in all we have the parameters α and β expressed in terms of the first two moments

$$\alpha = \frac{2M_2-M_1^2}{M_2-M_1^2} \tag{C.10}$$

$$\beta = \frac{M_2-M_1^2}{M_1M_2} \tag{C.11}$$

C.2.2 Density and derivatives

The density function g_R of the reciprocal gamma distribution is

$$\begin{aligned} g_R(y) &= \frac{1}{y^2}g(\frac{1}{y}) \\ &= \frac{1}{y^2}\Gamma(\alpha)^{-1}\beta^{-\alpha}(\frac{1}{y})^{\alpha-1}\exp(-\beta^{-1}\frac{1}{y}) \\ &= \Gamma(\alpha)^{-1}\beta^{-\alpha}y^{-\alpha-1}\exp(-\beta^{-1}y^{-1}), \quad y>0 \end{aligned} \tag{C.12}$$

where g is the density function of the gamma distribution, cf. (6.1). Set

$$\begin{aligned} c(\alpha,\beta) &= \Gamma(\alpha)^{-1}\beta^{-\alpha} && \text{(C.13)} \\ h(y) &= y^{-\alpha-1}\exp(-\beta^{-1}y^{-1}), \quad y>0 && \text{(C.14)} \end{aligned}$$

so

$$g_R(y) = c(\alpha,\beta)h(y) \tag{C.15}$$

The first derivative of h with respect to y is

$$\begin{aligned} \frac{dh}{dy}(y) &= y^{-\alpha-1}\exp(-\beta^{-1}y^{-1})\beta^{-1}y^{-2} + (-\alpha-1)y^{-\alpha-2}\exp(-\beta^{-1}y^{-1}) \\ &= \exp(-\beta^{-1}y^{-1})(\beta^{-1}y^{-\alpha-3} - (\alpha+1)y^{-\alpha-2}) \\ &= h(y)(\beta^{-1}y^{-2} - (\alpha+1)y^{-1}) && \text{(C.16)} \end{aligned}$$

The second derivative is then

$$\begin{aligned} \frac{d^2h}{dy^2}(y) &= h'(y)(\beta^{-1}y^{-2} - (\alpha+1)y^{-1}) \\ &\quad +h(y)(-2\beta^{-1}y^{-3} + (\alpha+1)y^{-2}) \\ &= h(y)(\beta^{-1}y^{-2} - (\alpha+1)y^{-1})^2 \\ &\quad +h(y)(-2\beta^{-1}y^{-3} + (\alpha+1)y^{-2}) \\ &= h(y)(\beta^{-2}y^{-4} + (\alpha+1)^2y^{-2} - 2\beta^{-1}y^{-2}(\alpha+1)y^{-1}) \\ &\quad +h(y)(-2\beta^{-1}y^{-3} + (\alpha+1)y^{-2}) \\ &= h(y)(\beta^{-2}y^{-4} - 2\beta^{-1}(\alpha+2)y^{-3} + (\alpha^2+3\alpha+2)y^{-2}) && \text{(C.17)} \end{aligned}$$

From the above, the first and second order derivatives of g_R, for $y>0$, are

$$\begin{aligned} \frac{dg_R}{dy}(y) &= c(\alpha,\beta)h'(y) \\ &= g_R(y)(\beta^{-1}y^{-2} - (\alpha+1)y^{-1}) \\ &= \Gamma(\alpha)^{-1}(\beta^{-\alpha-1}y^{-\alpha-3} - (\alpha+1)\beta^{-\alpha}y^{-\alpha-2}) \\ &\quad \cdot\exp(-\beta^{-1}y^{-1}) && \text{(C.18)} \end{aligned}$$

and

$$\begin{aligned} \frac{d^2g_R}{dy^2}(y) &= c(\alpha,\beta)h''(y) \\ &= g_R(y)(\beta^{-2}y^{-4} - 2\beta^{-1}(\alpha+2)y^{-3} + (\alpha^2+3\alpha+2)y^{-2}) \\ &= \Gamma(\alpha)^{-1}\{\beta^{-\alpha-2}y^{-\alpha-5} - 2\beta^{-\alpha-1}(\alpha+2)y^{-\alpha-4} \\ &\quad +(\alpha^2+3\alpha+2)\beta^{-\alpha}y^{-\alpha-3}\}\exp(-\beta^{-1}y^{-1}) && \text{(C.19)} \end{aligned}$$

C.3 Moments

Formulae for the first two moments of the arithmetic average are given in [Levy, 1992]. His average is

$$A_{Levy} = \frac{1}{n+1}\sum_{i=0}^{n} S_{t_i} \tag{C.20}$$

so we have to modify them to our specification of the average

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{C.21}$$

This is done as follows

$$\begin{aligned} A_{Levy} &= \frac{1}{n+1}\sum_{i=0}^{n} S_{t_i} = \frac{n}{n(n+1)}(S_0 + \sum_{i=1}^{n} S_{t_i}) \\ &= \frac{S_0}{n+1} + \frac{n}{n+1}\frac{1}{n}\sum_{i=1}^{n} S_{t_i} = \frac{S_0}{n+1} + \frac{n}{n+1}A \end{aligned} \tag{C.22}$$

Using (C.22), the first moment is

$$\begin{aligned} E[A_{Levy}] &= E[\frac{S_0}{n+1} + \frac{n}{n+1}A] \\ &= \frac{S_0}{n+1} + \frac{n}{n+1}E[A] \end{aligned} \tag{C.23}$$

and the second moment is

$$\begin{aligned} E[A_{Levy}^2] &= E[(\frac{S_0}{n+1} + \frac{n}{n+1}A)^2] \\ &= E[(\frac{S_0}{n+1})^2 + (\frac{n}{n+1})^2A^2 + 2\frac{S_0}{n+1}\frac{n}{n+1}A] \\ &= \frac{n^2}{(n+1)^2}E[A^2] + 2S_0\frac{n}{(n+1)^2}E[A] + \frac{S_0^2}{(n+1)^2} \end{aligned} \tag{C.24}$$

Inverting (C.23) and (C.24) gives

$$E[A] = \frac{1}{n}((n+1)E[A_{Levy}] - S_0) \tag{C.25}$$

and

$$\begin{aligned} E[A^2] &= \frac{1}{n^2}((n+1)^2E[A_{Levy}^2] - 2S_0nE[A] - S_0^2) \\ &= \frac{1}{n^2}((n+1)^2E[A_{Levy}^2] - 2S_0(n+1)E[A_{Levy}] + S_0^2) \end{aligned} \tag{C.26}$$

Levy's formulae with $\xi = 0,\ m = 0,\ t = 0,\ T_0 = 0$ (in his notation) are

$$E[A_{Levy}] = \frac{S_0}{n+1} + \frac{S_0}{n+1} e^{(r-q)h} \frac{1 - e^{(r-q)nh}}{1 - e^{(r-q)h}} \tag{C.27}$$

and

$$\begin{aligned} E[A^2_{Levy}] &= (\frac{S_0}{n+1})^2 + 2(\frac{S_0}{n+1})^2 e^{(r-q)h} \frac{1 - e^{(r-q)nh}}{1 - e^{(r-q)h}} \\ &\quad + (\frac{S_0}{n+1})^2 (a_1 - a_2 + a_3 - a_4) \\ &= \frac{S_0}{n+1}(2E[A_{Levy}] - \frac{S_0}{n+1}) + (\frac{S_0}{n+1})^2 (a_1 - a_2 + a_3 - a_4) \\ &= \frac{S_0}{n+1}(2E[A_{Levy}] + \frac{S_0}{n+1}(a_1 - a_2 + a_3 - a_4 - 1)) \qquad \text{(C.28)} \end{aligned}$$

where

$$a_1 = \frac{e^{(2(r-q)+\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q)h})(1 - e^{(2(r-q)+\sigma^2)h})} \tag{C.29}$$

$$a_2 = \frac{e^{((r-q)(n+2)+\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h})} \tag{C.30}$$

$$a_3 = \frac{e^{(3(r-q)+\sigma^2)h} - e^{((r-q)(n+2)+\sigma^2)h}}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h})} \tag{C.31}$$

$$a_4 = \frac{e^{(4(r-q)+2\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q+\sigma^2)h})(1 - e^{(2(r-q)+\sigma^2)h})} \tag{C.32}$$

The modified expressions for the moments are now obtained from (C.25) and (C.26)

$$E[A] = \frac{S_0}{n} e^{(r-q)h} \frac{1 - e^{(r-q)nh}}{1 - e^{(r-q)h}} \tag{C.33}$$

and

$$E[A^2] = \frac{S_0^2}{n^2}(a_1 - a_2 + a_3 - a_4) \tag{C.34}$$

where the a_i are given by (C.29)-(C.32).

C.4 Visual Basic code

```
Function MilevskyPosner(sigma As Double, K As Double, n As Integer, _
  q As Double, r As Double, S_0 As Double, T As Double) As Double
Dim sigSq As Double, h As Double, g As Double
Dim A1 As Double, A2 As Double, A3 As Double, A4 As Double
Dim M1 As Double, M2 As Double
Dim alpha As Double, beta As Double
Dim G1 As Double, G2 As Double
Dim price As Double

'sigma: volatility
'K:     strike price
'n:     # averaging times
'q:     stock dividend
'r:     interest rate
'S_0:   stock price at time 0
'T:     time to maturity

sigSq = sigma ^2
h = T / n   'time increment between averaging dates
g = r - q

'adjusted Levy formula for 1. and 2. moment of average
If g = 0 Then
  M1 = S_0
  'continuous time M2
  M2 = 2 * S_0 ^2 / T ^2 * (Exp(sigSq * T) - 1 - sigSq * T) / sigSq ^2
  Stop  'r = q (g = 0) not implemented in discrete time!
Else
  M1 = S_0 / n * Exp(g * h) * (1 - Exp(g * n * h)) / (1 - Exp(g * h))

  A1 = (1 - Exp(g * h)) * (1 - Exp((2 * g + sigSq) * h))
  A1 = (Exp((2 * g + sigSq) * h) - Exp((2 * g + sigSq) * _
       (n + 1) * h)) / A1
  A2 = (1 - Exp(g * h)) * (1 - Exp((g + sigSq) * h))
  A2 = (Exp((g * (n + 2) + sigSq) * h) - Exp((2 * g + sigSq) * _
       (n + 1) * h)) / A2
  A3 = (1 - Exp(g * h)) * (1 - Exp((g + sigSq) * h))
  A3 = (Exp((3 * g + sigSq) * h) - Exp((g * (n + 2) + sigSq) * h)) / A3
  A4 = (1 - Exp((g + sigSq) * h)) * (1 - Exp((2 * g + sigSq) * h))
```

```
  A4 = (Exp((2 * g + sigSq) * 2 * h) - Exp((2 * g + sigSq) * _
         (n + 1) * h)) / A4

  M2 = S_0 ^2 / n ^2 * (A1 - A2 + A3 - A4)
End If

'Momentmatching
alpha = (2 * M2 - M1 ^2) / (M2 - M1 ^2)
beta = (M2 - M1 ^2) / (M2 * M1)

G1 = WorksheetFunction.GammaDist(1 / K, alpha - 1, beta, True)
G2 = WorksheetFunction.GammaDist(1 / K, alpha, beta, True)

price = Exp(-r * T) * (M1 * G1 - K * G2)     'Asian option price

MilevskyPosner = price

End Function

Function MilevskyPosnerContinuous(sigma As Double, K As Double, _
  n As Integer, q As Double, r As Double, S_0 As Double, _
  T As Double) As Double
Dim sigSq As Double
Dim M1 As Double, M2 As Double, M2_sub As Double
Dim alpha As Double, beta As Double
Dim G1 As Double, G2 As Double
Dim price As Double

sigSq = sigma ^2
'The variable n is not used!

If r = q Then
  M1 = S_0
  M2 = 2 * S_0 ^2 / T ^2 * (Exp(sigSq * T) - 1 - sigSq * T) / sigSq ^2
Else
  M1 = S_0 * (Exp((r - q) * T) - 1)
  M1 = M1 / ((r - q) * T)

  M2_sub = (r - q + sigSq) * (2 * r - 2 * q + sigSq)
  M2_sub = Exp((2 * (r - q) + sigSq) * T) / M2_sub
```

```
  M2 = 1 / (2 * (r - q) + sigSq)
  M2 = M2 - Exp((r - q) * T) / (r - q + sigSq)
  M2 = M2 / (r - q) + M2_sub
  M2 = M2 * 2 * S_0 ^2 / T ^2
End If

'Momentmatching
alpha = (2 * M2 - M1 ^2) / (M2 - M1 ^2)
beta = (M2 - M1 ^2) / (M2 * M1)

G1 = WorksheetFunction.GammaDist(1 / K, alpha - 1, beta, True)
G2 = WorksheetFunction.GammaDist(1 / K, alpha, beta, True)

price = Exp(-r * T) * (M1 * G1 - K * G2)     'Asian option price

MilevskyPosnerContinuous = price

End Function

Function MPimproved(sigma As Double, K As Double, n As Integer, _
  q As Double, r As Double, S_0 As Double, T As Double) As Double
'Uses the TW method in a MP setup
Dim sigSq As Double, h As Double, k2 As Double
Dim R1 As Double, R2 As Double, R3 As Double, R4 As Double
Dim L1 As Double, L2 As Double, L3 As Double, L4 As Double
Dim j As Integer
Dim Y1 As Double, Y2 As Double, Y3 As Double, Y4 As Double
Dim A1 As Double, A2 As Double, A3 As Double, A4 As Double
Dim alfa As Double, beta As Double
Dim c2 As Double, c3 As Double, c4 As Double
Dim G1 As Double, G2 As Double
Dim price As Double

'sigma: volatility
'K:     strike price
'n:     # averaging times
'q:     stock dividend
'r:     interest rate
'S_0:   stock price at time 0
'T:     time to maturity
```

```
sigSq = sigma ^2
h = T / n           'time increment between averaging dates
k2 = n * K / S_0    '(lower case) k in TW article

'PART 1 - calculate moments of true distribution
'i=T-(n-1) in TW article
R1 = LogMoment(1, (r - q - sigSq / 2) * h, sigSq * h)
R2 = LogMoment(2, (r - q - sigSq / 2) * h, sigSq * h)
R3 = LogMoment(3, (r - q - sigSq / 2) * h, sigSq * h)
R4 = LogMoment(4, (r - q - sigSq / 2) * h, sigSq * h)
L1 = 1      'L(T+1)=L(i+n)=1
L2 = 1
L3 = 1
L4 = 1

For j = (n - 1) To 1 Step -1    'time i+j, i.e. time T to time T-n+2
  'Priority L4,L3,L2,L1 important!
  L4 = 1 + R4 * L4 + 4 * R3 * L3 + 6 * R2 * L2 + 4 * R1 * L1
  L3 = 1 + R3 * L3 + 3 * R2 * L2 + 3 * R1 * L1
  L2 = 1 + R2 * L2 + 2 * R1 * L1
  L1 = 1 + R1 * L1
Next j

'Moments of true distribution
Y1 = R1 * L1
Y2 = R2 * L2
Y3 = R3 * L3
Y4 = R4 * L4

'PART 2 - calculate moments of approximating distribution
'match moments of approx. dist. to true dist.
alfa = (2 * Y2 - Y1 ^2) / (Y2 - Y1 ^2)
beta = (Y2 - Y1 ^2) / (Y1 * Y2)

'Moments of approximating distribution
A1 = Y1         'per construction
A2 = Y2         'per construction
A3 = RGMoment(3, alfa, beta)
A4 = RGMoment(4, alfa, beta)

'PART 3 - calculate option price
```

```
c2 = Y2 - Y1 ^2 - (A2 - A1 ^2)
c3 = Y3 + 2 * Y1 ^3 - 3 * Y2 * Y1
c3 = c3 - (A3 + 2 * A1 ^3 - 3 * A2 * A1)
c4 = Y4 - 6 * Y1 ^4 - 4 * Y3 * Y1 + 12 * Y2 * Y1 ^2 - 3 * Y2 ^2
c4 = c4 - (A4 - 6 * A1 ^4 - 4 * A3 * A1 + 12 * A2 * A1 ^2 - 3 * A2 ^2)
c4 = c4 + 3 * c2 ^2

G1 = WorksheetFunction.GammaDist(1 / k2, alfa - 1, beta, True)
G2 = WorksheetFunction.GammaDist(1 / k2, alfa, beta, True)

price = Y1 * G1 - k2 * G2
price = price + c2 / 2 * RGDensity(0, k2, alfa, beta)
price = price - c3 / 6 * RGDensity(1, k2, alfa, beta)
price = price + c4 / 24 * RGDensity(2, k2, alfa, beta)
price = price * Exp(-r * T) * S_0 / n

MPimproved = price

End Function

Function RGMoment(m As Integer, alfa As Double, _
  beta As Double) As Double
'Returns the m'th moment of reciprocal gamma(alfa, beta)

If alfa > m Then
 If m = 1 Then
  RGMoment = 1 / (beta * (alfa - 1))
 ElseIf m = 2 Then
  RGMoment = 1 / (beta ^2 * (alfa - 1) * (alfa - 2))
 ElseIf m = 3 Then
  RGMoment = 1 / (beta ^3 * (alfa - 1) * (alfa - 2) * (alfa - 3))
 ElseIf m = 4 Then
  RGMoment = 1 / (beta ^4 * (alfa - 1) * (alfa - 2) * _
    (alfa - 3) * (alfa - 4))
 Else
  Stop
 End If
Else
 Stop
End If
```

```
End Function

Function RGDensity(deriv As Integer, x As Double, _
  alfa As Double, beta As Double) As Double
'Density function (and derivatives) of reciprocal gamma(alfa, beta)

Dim f As Double

f = beta ^(-alfa) * x ^(-alfa - 1) * Exp(-1 / (beta * x)) _
  / Exp(WorksheetFunction.GammaLn(alfa))

If deriv = 0 Then
  'reciprocal gamma density f(x)
  RGDensity = f
ElseIf deriv = 1 Then
  '1. order derivative f'(x)
  RGDensity = f * (beta ^(-1) * x ^(-2) - (alfa + 1) * x ^(-1))
ElseIf deriv = 2 Then
  '2. order derivative f''(x)
  RGDensity = f * (beta ^(-2) * x ^(-4) - 2 * beta ^(-1) * _
    (alfa + 2) * x ^(-3) + (alfa ^2 + 3 * alfa + 2) * x ^(-2))
Else
  Stop
End If

End Function
```

Appendix D

Vorst

D.1 Visual Basic code

```
Function Vorst92(sigma As Double, K As Double, n As Integer, _
  q As Double, r As Double, S_0 As Double, T As Double) As Variant
Dim sigSq As Double, h As Double
Dim meanGeo As Double, varGeo As Double, EA As Double, EG As Double
Dim d1 As Double, d2 As Double, priceGeo As Double
Dim upBound As Double
Dim Knew As Double, d1new As Double, d2new As Double, price As Double
Dim output(2) As Double

'sigma: volatility
'K:     strike price
'n:     # averaging times
'q:     stock dividend
'r:     interest rate
'S_0:   stock price at time 0
'T:     time to maturity

sigSq = sigma ^2
h = T / n   'time increment between averaging dates

'Mean and variance of logarithm of geometric average
meanGeo = Log(S_0) + (r - q - sigSq / 2) * (T + h) / 2
varGeo = sigSq * (h + (T - h) * (2 * n - 1) / (6 * n))

'Mean of arithmetic and geometric averages
```

```
EA = S_0 / n * Exp((r - q) * h) * (1 - Exp((r - q) * n * h)) _
  / (1 - Exp((r - q) * h))
EG = Exp(meanGeo + varGeo / 2)

'Price of geometric Asian option
d1 = (meanGeo - Log(K) + varGeo) / Sqr(varGeo)
d2 = d1 - Sqr(varGeo)
priceGeo = Exp(-r * T) * (EG * WorksheetFunction.NormSDist(d1) - _
  K * WorksheetFunction.NormSDist(d2))

upBound = priceGeo + Exp(-r * T) * (EA - EG)

'Approximate price of Asian option [Vorst, 1992]
Knew = K - (EA - EG)    'Adjusted strike price
d1new = (meanGeo - Log(Knew) + varGeo) / Sqr(varGeo)
d2new = d1new - Sqr(varGeo)
price = Exp(-r * T) * (EG * WorksheetFunction.NormSDist(d1new) - _
  Knew * WorksheetFunction.NormSDist(d2new))

output(0) = price
output(1) = upBound     'upper bound
output(2) = priceGeo    'lower bound

Vorst92 = output

End Function
```

Appendix E

Curran

E.1 Visual Basic code

```
Function Curran94(sigma As Double, K As Double, n As Integer, _
  q As Double, r As Double, S_0 As Double, T As Double) As Variant
Dim sigSq As Double, h As Double
Dim meanG As Double, varG As Double
Dim i As Integer
Dim meanS() As Double, varS() As Double, covSG() As Double
Dim d1() As Double, d2 As Double
Dim I1 As Double, I2 As Double, c2 As Double
Dim y As Double, EAG As Double, delta As Double
Dim price As Double
Dim output(1) As Double

'sigma: volatility
'K:     strike price
'n:     # averaging times
'q:     stock dividend
'r:     interest rate
'S_0:   stock price at time 0
'T:     time to maturity

ReDim meanS(n)
ReDim varS(n)
ReDim covSG(n)
ReDim d1(n)
```

```
sigSq = sigma ^2
h = T / n   'time increment between averaging dates

'Mean and variance of logarithm of geometric average ln(G)
meanG = Log(S_0) + (r - q - sigSq / 2) * (T + h) / 2
varG = sigSq * h * (2 * n + 1) * (n + 1) / (6 * n)

For i = 1 To n
  'Mean and variance of logarithm of stock price ln(S)
  meanS(i) = Log(S_0) + (r - q - sigSq / 2) * i * h
  varS(i) = sigSq * i * h
  'Covariance between ln(S) and ln(G)
  covSG(i) = sigSq * h * ((2 * n + 1) * i - i ^2) / (2 * n)
Next i

'Calculate L to replace K as the lower limit in the integral
delta = 10 ^-6     'stepsize
y = Log(K)         'initial value (which is too high)
EAG = CondMean(y, sigma, K, n, q, r, S_0, T)

Do While EAG > K
  y = y - delta
  EAG = CondMean(y, sigma, K, n, q, r, S_0, T)
Loop

'At this point EAG < K so we use the last but one y
y = y + delta
L = Exp(y)

'Calculate option price
I1 = 0
For i = 1 To n
  d1(i) = (meanG - Log(L) + covSG(i)) / Sqr(varG)
  I1 = I1 + Exp(meanS(i) + varS(i) / 2) _
    * WorksheetFunction.NormSDist(d1(i))
Next i
I1 = I1 / n

d2 = (meanG - Log(L)) / Sqr(varG)
I2 = K * WorksheetFunction.NormSDist(d2)
```

```
price = Exp(-r * T) * (I1 - I2)

output(0) = price
output(1) = L

Curran94 = output()

End Function

Function CondMean(y As Double, sigma As Double, K As Double, n As Integer, _
  q As Double, r As Double, S_0 As Double, T As Double) As Double
Dim sigSq As Double, h As Double
Dim meanG As Double, varG As Double
Dim i As Integer
Dim meanS() As Double, varS() As Double, covSG() As Double
Dim result As Double

ReDim meanS(n)
ReDim varS(n)
ReDim covSG(n)

sigSq = sigma ^2
h = T / n   'time increment between averaging dates

'Mean and variance of logarithm of geometric average ln(G)
meanG = Log(S_0) + (r - q - sigSq / 2) * (T + h) / 2
varG = sigSq * h * (2 * n + 1) * (n + 1) / (6 * n)

For i = 1 To n
  'Mean and variance of logarithm of stock price ln(S)
  meanS(i) = Log(S_0) + (r - q - sigSq / 2) * i * h
  varS(i) = sigSq * i * h
  'Covariance between ln(S) and ln(G)
  covSG(i) = sigSq * h * ((2 * n + 1) * i - i ^2) / (2 * n)
Next i

result = 0
For i = 1 To n
  result = result + Exp(meanS(i) + (y - meanG) * covSG(i) / varG _
           + (varS(i) - covSG(i) ^2 / varG) / 2)
Next i
```

```
result = result / n

CondMean = result

End Function
```

Bibliography

[Baxter and Rennie, 1996] Baxter, M. and Rennie, A. (1996). *Financial Calculus*. Cambridge University Press.

[Black and Scholes, 1973] Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654.

[Bouaziz et al., 1994] Bouaziz, L., Briys, E., and Crouhy, M. (1994). The pricing of forward-starting asian options. *Journal of Banking and Finance*, 18:823–839.

[Curran, 1994] Curran, M. (1994). Valuing asian and portfolio options by conditioning on the geometric mean price. *Management Science*, 40:1705–1711.

[Dewynne and Wilmott, 1995a] Dewynne, J. N. and Wilmott, P. (1995a). Asian options as linear complementarity problems: Analysis and finite-difference solutions. *Advances in Futures and Options Research*, 8:145–173.

[Dewynne and Wilmott, 1995b] Dewynne, J. N. and Wilmott, P. (1995b). A note on average rate options with discrete sampling. *Journal of Applied Mathematics*, 55(1):267–276.

[Duffie, 1996] Duffie, D. (1996). *Dynamic Asset Pricing*. Princeton University Press.

[Glasserman, 1998] Glasserman, P. (1998). Monte carlo methods for option pricing. Ph.D. course lecture notes.

[Hoffmann-Jørgensen, 1994a] Hoffmann-Jørgensen, J. (1994a). *Probability with a View Toward Statistics*, volume I. Chapman and Hall.

[Hoffmann-Jørgensen, 1994b] Hoffmann-Jørgensen, J. (1994b). *Probability with a View Toward Statistics*, volume II. Chapman and Hall.

[Hull, 1997] Hull, J. C. (1997). *Options, Futures, and Other Derivatives.* Prentice-Hall, Inc., third edition.

[Jarrow and Rudd, 1982] Jarrow, R. and Rudd, A. (1982). Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics*, 10:347–369.

[Johnson et al., 1994] Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, volume 1. Wiley, second edition.

[Karatzas and Shreve, 1991] Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus.* Springer, second edition.

[Kemna and Vorst, 1990] Kemna, A. G. Z. and Vorst, A. C. F. (1990). A pricing method for options based on average asset values. *Journal of Banking and Finance*, 14:113–129.

[Leeson, 1996] Leeson, N. (1996). *Rogue Trader.* Little Brown.

[Levy, 1992] Levy, E. (1992). Pricing european average rate currency options. *Journal of International Money and Finance*, 11:474–491.

[Milevsky and Posner, 1998] Milevsky, M. A. and Posner, S. E. (1998). Asian options, the sum of lognormals, and the reciprocal gamma distribution. *Journal of Financial and Quantitative Analysis*, 33(3):409–422.

[Press et al., 1992] Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T. (1992). *Numerical Recipes in C.* Cambridge, second edition.

[Rogers and Shi, 1995] Rogers, L. and Shi, Z. (1995). The value of an asian option. *Journal of Applied Probability*, 32:1077–1088.

[Turnbull and Wakeman, 1991] Turnbull, S. M. and Wakeman, L. M. (1991). A quick algorithm for pricing european average options. *Journal of Financial and Quantitative Analysis*, 26(3):377–389.

[Vorst, 1992] Vorst, T. (1992). Prices and hedge ratios of average exchange rate options. *International Review of Financial Analysis*, 1(3):179–193.